Autoparallels in post-Riemannian spacetimes

Jens Boos
jens.boos@kit.edu
Karlsruhe Institute of Technology, Germany

Based on: J. Boos, arXiv:2504.06013 + arXiv:2505.XXXXX

Jiří Bičák Memorial Conference
Univerzita Karlova, Praha, Česko
Thursday, May 29, 2025, 2:20pm

1/5 Motivation

Current weak-field constraints on torsion

Constraints on axial torsion (Lämmerzahl 1997 via Hughes–Drever experiments):

$$T_{\text{axial}} \sim |T_{[\mu\nu\rho]}| \leq 1.5 \times 10^{-15}\text{m}^{-1}$$

This is a rather weak constraint. By dimensional analysis, at the surface of the earth:

$$\boxed{\begin{aligned} T &\sim \frac{GM_{\text{earth}}}{c^2 R_{\text{earth}}^2} \sim 1.1 \times 10^{-16}\text{m}^{-1} \\ R &\sim \frac{GM_{\text{earth}}}{c^2 R_{\text{earth}}^3} \sim 1.7 \times 10^{-23}\text{m}^{-2} \end{aligned}}$$

$$\boxed{\begin{aligned} T &\sim \frac{c^2}{GM_\odot} \sim 6.8 \times 10^{-4}\text{m}^{-1} \\ R &\sim \frac{c^4}{(GM_\odot)^2} \sim 4.6 \times 10^{-7}\text{m}^{-2} \end{aligned}}$$

Astrophysical sources can be much more compact, hence **larger values possible**!

How does the presence of such post-Riemannian quantities affect the motion of particles?

2/5 Framework

Lagrangian of (parity-even) Poincaré gauge gravity

Hehl et al (1976)
Obukhov (2023)

$$\mathcal{L} = \frac{1}{2\kappa}\left[\underbrace{\boxed{a_0 R - 2\Lambda}}_{\text{Einstein sector}} + \underbrace{\boxed{\frac{1}{2}\sum_{I=1}^{3} a_I \left({}^{(I)}T_{\mu\nu}{}^{\rho}\right)^2}}_{\text{teleparallel sector}} + \underbrace{\boxed{\frac{\ell_p^2}{2}\sum_{I=1}^{6} b_I \left({}^{(I)}R_{\mu\nu\rho\sigma}\right)^2}}_{\text{strong gravity sector}} \right]$$

All 10 couplings a_I and b_I are dimensionless, $\kappa =$ gravitational constant, $\ell_p =$ Planck length.

Field equations of Poincaré gauge gravity

Obukhov (2023)

$$a_0\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\right)+\Lambda g_{\mu\nu}+q^{\mathrm{T}}_{\mu\nu}+\ell^2_{\mathrm{Pl}}\,q^{\mathrm{R}}_{\mu\nu}-(\nabla_\alpha+T_\alpha)h_\nu{}^\alpha{}_\mu-\frac{1}{2}T_{\alpha\beta\nu}h^{\alpha\beta}{}_\mu=\kappa T_{\mu\nu}\,,$$

$$a_0(T_{\mu\nu}{}^\lambda+T_\mu\delta^\lambda_\nu-T_\nu\delta^\lambda_\mu)-h^\lambda{}_{\mu\nu}+h^\lambda{}_{\nu\mu}-2\ell^2_{\mathrm{Pl}}\left[(\nabla_\alpha-T_\alpha)h^{\lambda\alpha}{}_{\mu\nu}+\frac{1}{2}T_{\alpha\beta}{}^\lambda h^{\alpha\beta}{}_{\mu\nu}\right]=\kappa S_{\mu\nu}{}^\lambda\,,$$

Colors: **Einstein**, **teleparallel**, **strong gravity**. Auxiliary excitations h and traceless tensors q:

$$h^{\mu\nu}{}_\lambda=\sum_{I=1}^{3}a_I{}^{(I)}T^{\mu\nu}{}_\lambda\,,\qquad q^{\mathrm{T}}_{\mu\nu}=T_{\mu\alpha\beta}h_\nu{}^{\alpha\beta}-\frac{1}{4}g_{\mu\nu}T_{\alpha\beta}{}^\gamma h^{\alpha\beta}{}_\gamma\,,$$

$$h^{\mu\nu}{}_{\rho\sigma}=\sum_{b=1}^{6}b_I{}^{(6)}R^{\mu\nu}{}_{\rho\sigma}\,,\qquad q^{\mathrm{R}}_{\mu\nu}=R_{\mu\alpha}{}^{\beta\gamma}h_\nu{}^\alpha{}_{\beta\gamma}-\frac{1}{4}g_{\mu\nu}R_{\alpha\beta}{}^{\gamma\delta}h^{\alpha\beta}{}_{\gamma\delta}\,.$$

The two matter currents are the energy-momentum $T_{\mu\nu}$ and the spin-angular momentum $S_{\mu\nu}{}^\lambda$.

Particle motion in the presence of spacetime torsion

Affine connection is a sum of **Levi-Civita part** and **torsion part**:

$$\Gamma^{\lambda}{}_{\mu\nu} = \widetilde{\Gamma}^{\lambda}{}_{\mu\nu} + K^{\lambda}{}_{\mu\nu} = \boxed{\frac{1}{2} g^{\lambda\alpha} \left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right)} + \boxed{\frac{1}{2}\left(T_{\mu\nu}{}^{\lambda} - T_{\mu}{}^{\lambda}{}_{\nu} - T_{\nu}{}^{\lambda}{}_{\mu}\right)} \, .$$

Particle motion in the presence of spacetime torsion

Affine connection is a sum of **Levi-Civita part** and **torsion part**:

$$\Gamma^{\lambda}{}_{\mu\nu} = \widetilde{\Gamma}^{\lambda}{}_{\mu\nu} + K^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha} \left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu} \right) + \frac{1}{2} \left(T_{\mu\nu}{}^{\lambda} - T_{\mu}{}^{\lambda}{}_{\nu} - T_{\nu}{}^{\lambda}{}_{\mu} \right) .$$

There are now two inequivalent ways to describe particle motion locally:

$$u^\alpha \widetilde{\nabla}_\alpha u^\mu = 0 \qquad\qquad u^\alpha \nabla_\alpha u^\mu = 0$$

geodesics — "shortest paths"

autoparallels — "straightest paths"

Particle motion in the presence of spacetime torsion

Affine connection is a sum of **Levi-Civita part** and **torsion part**:

$$\Gamma^{\lambda}{}_{\mu\nu} = \widetilde{\Gamma}^{\lambda}{}_{\mu\nu} + K^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha} \left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu} \right) + \frac{1}{2} \left(T_{\mu\nu}{}^{\lambda} - T_{\mu}{}^{\lambda}{}_{\nu} - T_{\nu}{}^{\lambda}{}_{\mu} \right) .$$

There are now two inequivalent ways to describe particle motion locally:

$$u^\alpha \widetilde{\nabla}_\alpha u^\mu = 0$$

geodesics

"shortest paths"

$$u^\alpha \nabla_\alpha u^\mu = u^\alpha \widetilde{\nabla}_\alpha u^\mu - T_\alpha{}^\mu{}_\beta u^\alpha u^\beta = 0$$

autoparallels

"straightest paths"

Particle motion in the presence of spacetime torsion

Affine connection is a sum of **Levi-Civita part** and **torsion part**:

$$\Gamma^{\lambda}{}_{\mu\nu} = \widetilde{\Gamma}^{\lambda}{}_{\mu\nu} + K^{\lambda}{}_{\mu\nu} = \boxed{\frac{1}{2} g^{\lambda\alpha} \left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu} \right)} + \boxed{\frac{1}{2} \left(T_{\mu\nu}{}^{\lambda} - T_{\mu}{}^{\lambda}{}_{\nu} - T_{\nu}{}^{\lambda}{}_{\mu} \right)} \,.$$

There are now two inequivalent ways to describe particle motion locally:

$$u^\alpha \widetilde{\nabla}_\alpha u^\mu = 0$$

geodesics

"shortest paths"

$$u^\alpha \nabla_\alpha u^\mu = u^\alpha \widetilde{\nabla}_\alpha u^\mu - \boxed{T_{\alpha}{}^{\mu}{}_{\beta} u^\alpha u^\beta} = 0$$

autoparallels

"straightest paths"

Today, I want to focus on autoparallels around a **black hole** with a **torsion profile**.

Example: autoparallels in flat 2D spacetime $T_{tx}{}^{\mu} = (\tau, \chi)$

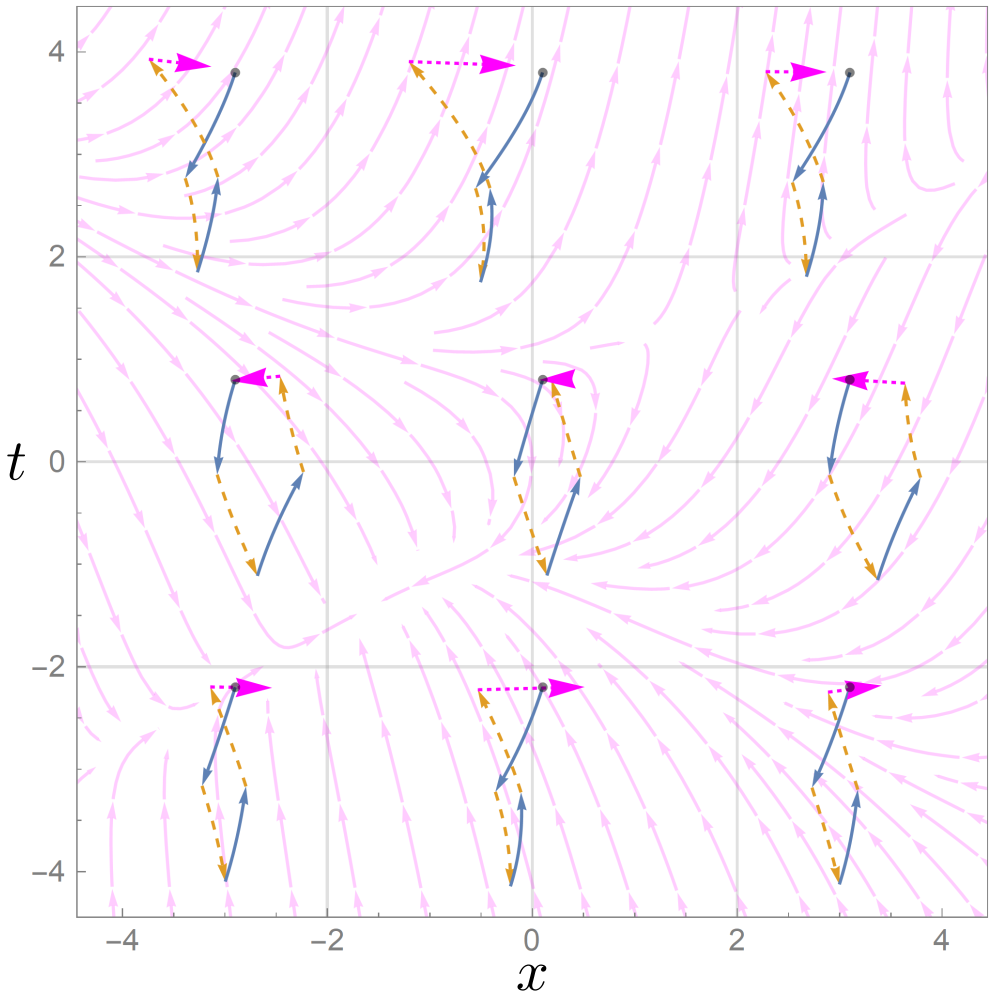
A bit like a Lorentz force:

$$\begin{aligned} \ddot{t} &= +(\chi \dot{x} - \tau \dot{t})\dot{x}\,, \\ \ddot{x} &= -(\tau \dot{t} - \chi \dot{x})\dot{t}\,. \end{aligned}$$

Example: autoparallels in flat 2D spacetime $T_{tx}{}^{\mu} = (\tau, \chi)$

A bit like a Lorentz force:

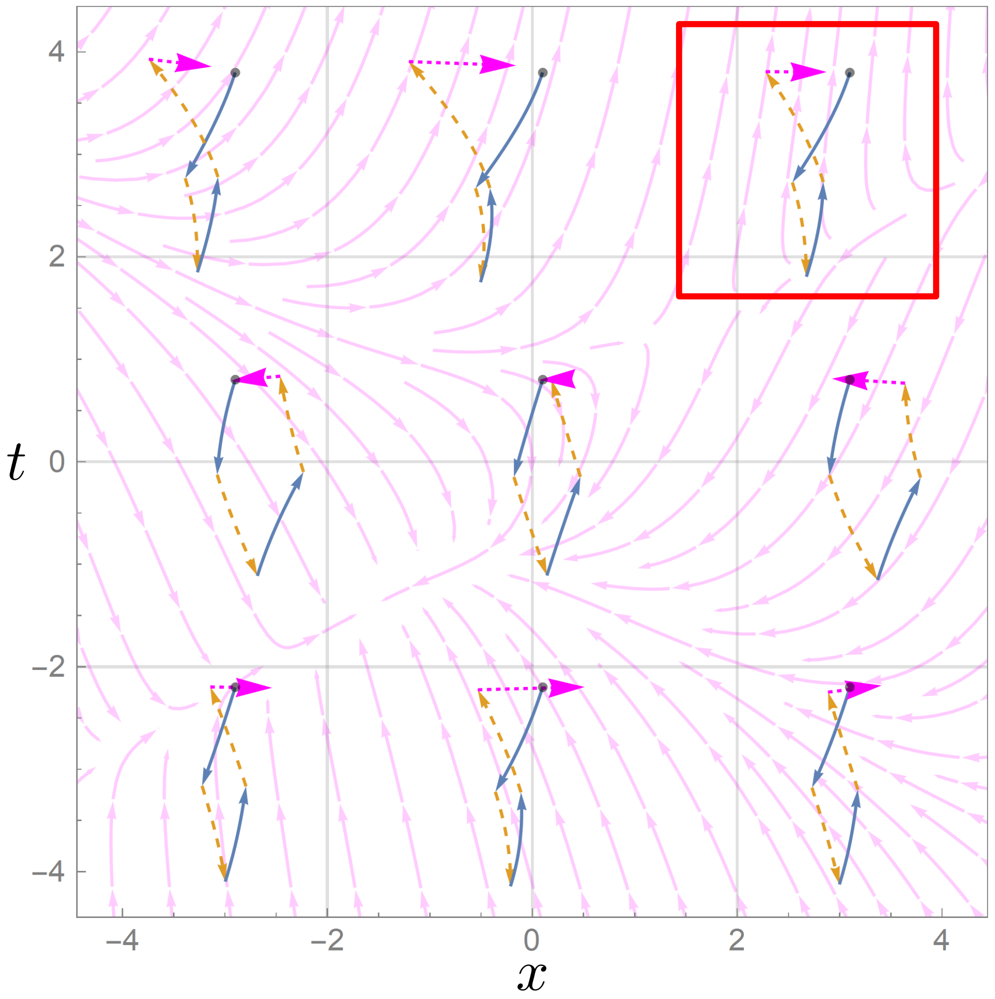
$$\ddot{t} = +(\chi\dot{x} - \tau\dot{t})\dot{x}\,,$$
$$\ddot{x} = -(\tau\dot{t} - \chi\dot{x})\dot{t}\,.$$



Example: autoparallels in flat 2D spacetime $T_{tx}{}^{\mu} = (\tau, \chi)$

A bit like a Lorentz force:

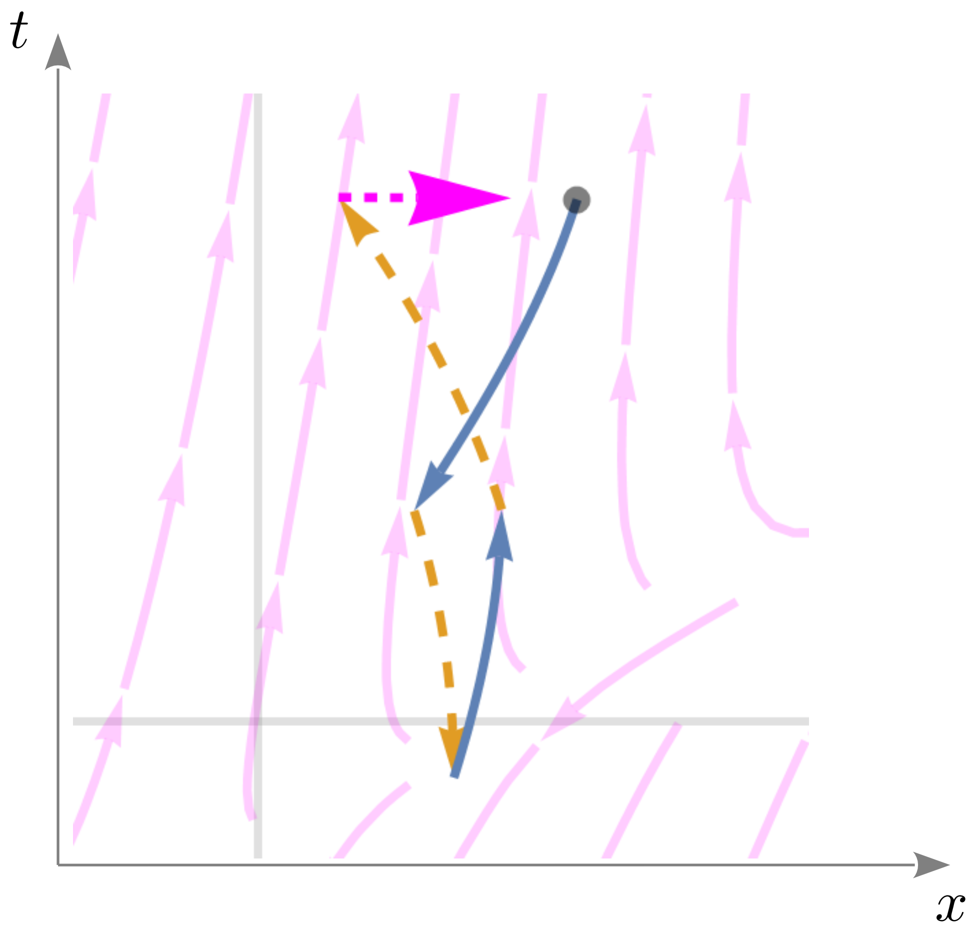
$$\ddot{t} = +(\chi\dot{x} - \tau\dot{t})\dot{x}\,,$$
$$\ddot{x} = -(\tau\dot{t} - \chi\dot{x})\dot{t}\,.$$



Example: autoparallels in flat 2D spacetime $T_{tx}{}^{\mu} = (\tau, \chi)$

A bit like a Lorentz force:

$$\ddot{t} = +(\chi \dot{x} - \tau \dot{t})\dot{x}\,,$$
$$\ddot{x} = -(\tau \dot{t} - \chi \dot{x})\dot{t}\,.$$



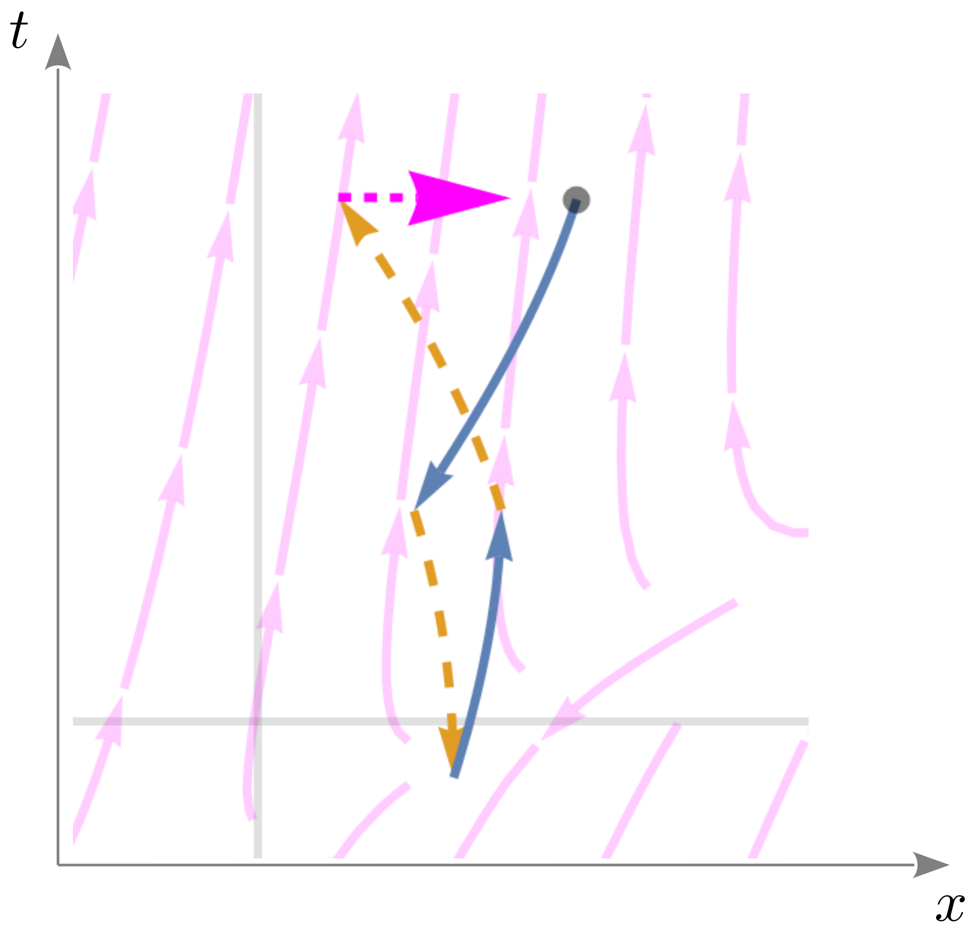
Example: autoparallels in flat 2D spacetime $T_{tx}{}^{\mu} = (\tau, \chi)$

A bit like a Lorentz force:

$$\ddot{t} = +(\chi\dot{x} - \tau\dot{t})\dot{x}\,,$$
$$\ddot{x} = -(\tau\dot{t} - \chi\dot{x})\dot{t}\,.$$

Torsion is a **closure failure**:

$$[\nabla_\mu, \nabla_\nu]\, V^\rho = R_{\mu\nu}{}^{\rho}{}_{\alpha} V^\alpha$$
$$\qquad - T_{\mu\nu}{}^{\alpha} \nabla_\alpha V^\rho$$



3/5 Schwarzschild with torsion

Schwarzschild black hole with GM/r^2-torsion profile

Baekler[†] (1981)

Schwarzschild
(identical to GR)

$$\boxed{\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega^2\,,} \qquad f(r) = 1 - \frac{2GM}{r} - \frac{\Lambda_{\text{eff}}}{3}r^2\,,$$

$$T_{tr}{}^{t} = T_{r\theta}{}^{\theta} = T_{r\varphi}{}^{\varphi} = \boxed{-\frac{GM}{r^2}\frac{1}{f(r)}}\,, \qquad T_{tr}{}^{r} = -T_{t\theta}{}^{\theta} = -T_{t\varphi}{}^{\varphi} = \boxed{-\frac{GM}{r^2}}\,.$$

supplemented by a torsion profile that scales as GM/r^2

Geometric properties

$$\boxed{\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega^2\,,} \qquad f(r) = 1 - \frac{2GM}{r} - \frac{\Lambda_{\text{eff}}}{3}r^2\,,$$

$$T_{tr}{}^{t} = T_{r\theta}{}^{\theta} = T_{r\varphi}{}^{\varphi} = \boxed{-\frac{GM}{r^2}\frac{1}{f(r)}}\,, \qquad T_{tr}{}^{r} = -T_{t\theta}{}^{\theta} = -T_{t\varphi}{}^{\varphi} = \boxed{-\frac{GM}{r^2}}\,.$$

Effective cosmological constant $\Lambda_{\text{eff}} = -\frac{3(a_0+a_1)}{2(b_4+b_6)\ell_p^2}$.

Riemann–Cartan curvature: $R_{\mu\nu\rho\sigma} \propto (a_0 + a_1)$

Riemann curvature: unchanged compared to general relativity

Torsion: $(T_{\mu\nu}{}^{\rho})^2 = 0$

In the following: set $a_0 + a_1 = 0$. Flat (teleparallel) solution. GR sector: asymptotically flat.

Autoparallel Killing vectors

Sharif & Majeed (2009)
Petersen & Bonder (2019)
Boos (2025)

The autoparallel Killing equation $\nabla_\mu V_\nu + \nabla_\nu V_\mu = 0$ admits this maximal set of solutions:

Autoparallel Killing vectors

Sharif & Majeed (2009)
Petersen & Bonder (2019)
Boos (2025)

The autoparallel Killing equation $\nabla_\mu V_\nu + \nabla_\nu V_\mu = 0$ admits this maximal set of solutions:

$$\boxed{\begin{aligned}
\xi &= \frac{r-GM}{r-2GM}\,\partial_t + \frac{GM}{r}\,\partial_r\,,\\
\rho_1 &= \sin\theta\cos\varphi\,\left[\frac{GM}{r-2GM}\,\partial_t + \left(1-\frac{GM}{r}\right)\,\partial_r\right] + \frac{1}{r}\cos\theta\cos\varphi\,\partial_\theta - \frac{1}{r}\frac{\sin\phi}{\sin\theta}\,\partial_\varphi\,,\\
\rho_2 &= \sin\theta\sin\varphi\,\left[\frac{GM}{r-2GM}\,\partial_t + \left(1-\frac{GM}{r}\right)\,\partial_r\right] + \frac{1}{r}\cos\theta\sin\varphi\,\partial_\theta + \frac{1}{r}\frac{\cos\phi}{\sin\theta}\,\partial_\varphi\,,\\
\rho_3 &= \frac{GM}{r-2GM}\cos\theta\,\partial_t + \left(1-\frac{GM}{r}\right)\cos\theta\,\partial_r - \frac{1}{r}\sin\theta\,\partial_\theta\,.
\end{aligned}}$$

Autoparallel Killing vectors

Sharif & Majeed (2009)
Petersen & Bonder (2019)
Boos (2025)

The autoparallel Killing equation $\nabla_\mu V_\nu + \nabla_\nu V_\mu = 0$ admits this maximal set of solutions:

$$\boxed{\begin{aligned}
\xi &= \frac{r - GM}{r - 2GM}\,\partial_t + \frac{GM}{r}\,\partial_r\,, \\
\rho_1 &= \sin\theta\cos\varphi\left[\frac{GM}{r-2GM}\,\partial_t + \left(1 - \frac{GM}{r}\right)\,\partial_r\right] + \frac{1}{r}\cos\theta\cos\varphi\,\partial_\theta - \frac{1}{r}\frac{\sin\phi}{\sin\theta}\,\partial_\varphi\,, \\
\rho_2 &= \sin\theta\sin\varphi\left[\frac{GM}{r-2GM}\,\partial_t + \left(1 - \frac{GM}{r}\right)\,\partial_r\right] + \frac{1}{r}\cos\theta\sin\varphi\,\partial_\theta + \frac{1}{r}\frac{\cos\phi}{\sin\theta}\,\partial_\varphi\,, \\
\rho_3 &= \frac{GM}{r-2GM}\cos\theta\,\partial_t + \left(1 - \frac{GM}{r}\right)\cos\theta\,\partial_r - \frac{1}{r}\sin\theta\,\partial_\theta\,.
\end{aligned}}$$

Normalization $1 = -\xi\cdot\xi = \rho_1\cdot\rho_1 = \rho_2\cdot\rho_2 = \rho_3\cdot\rho_3$ and algebra $[\xi,\rho_I]_T = [\rho_I,\rho_J]_T = 0\,.$

$$[X,Y]^{\mu}_{\mathrm{T}} \equiv [X,Y]^{\mu} + T_{\alpha\beta}{}^{\mu} \boldsymbol{X}^{\alpha} \boldsymbol{Y}^{\beta}$$

Autoparallel Killing vectors

Sharif & Majeed (2009)
Petersen & Bonder (2019)
Boos (2025)

The autoparallel Killing equation $\nabla_\mu V_\nu + \nabla_\nu V_\mu = 0$ admits this maximal set of solutions:

$$\xi = \frac{r-GM}{r-2GM}\,\partial_t + \frac{GM}{r}\,\partial_r\,,$$

$$\rho_1 = \sin\theta\cos\varphi\left[\frac{GM}{r-2GM}\,\partial_t + \left(1-\frac{GM}{r}\right)\partial_r\right] + \frac{1}{r}\cos\theta\cos\varphi\,\partial_\theta - \frac{1}{r}\frac{\sin\phi}{\sin\theta}\,\partial_\varphi\,,$$

$$\rho_2 = \sin\theta\sin\varphi\left[\frac{GM}{r-2GM}\,\partial_t + \left(1-\frac{GM}{r}\right)\partial_r\right] + \frac{1}{r}\cos\theta\sin\varphi\,\partial_\theta + \frac{1}{r}\frac{\cos\phi}{\sin\theta}\,\partial_\varphi\,,$$

$$\rho_3 = \frac{GM}{r-2GM}\cos\theta\,\partial_t + \left(1-\frac{GM}{r}\right)\cos\theta\,\partial_r - \frac{1}{r}\sin\theta\,\partial_\theta\,.$$

Normalization $1 = -\xi\cdot\xi = \rho_1\cdot\rho_1 = \rho_2\cdot\rho_2 = \rho_3\cdot\rho_3$ and algebra $[\xi,\rho_I]_T = [\rho_I,\rho_J]_T = 0\,.$

Autoparallel Killing vectors

Sharif & Majeed (2009)
Petersen & Bonder (2019)
Boos (2025)

The autoparallel Killing equation $\nabla_\mu V_\nu + \nabla_\nu V_\mu = 0$ admits this maximal set of solutions:

$$\boxed{\begin{aligned}
\xi &= \frac{r-GM}{r-2GM}\,\partial_t + \frac{GM}{r}\,\partial_r\,, \\
\rho_1 &= \sin\theta\cos\varphi\left[\frac{GM}{r-2GM}\,\partial_t + \left(1-\frac{GM}{r}\right)\partial_r\right] + \frac{1}{r}\cos\theta\cos\varphi\,\partial_\theta - \frac{1}{r}\frac{\sin\phi}{\sin\theta}\,\partial_\varphi\,, \\
\rho_2 &= \sin\theta\sin\varphi\left[\frac{GM}{r-2GM}\,\partial_t + \left(1-\frac{GM}{r}\right)\partial_r\right] + \frac{1}{r}\cos\theta\sin\varphi\,\partial_\theta + \frac{1}{r}\frac{\cos\phi}{\sin\theta}\,\partial_\varphi\,, \\
\rho_3 &= \frac{GM}{r-2GM}\cos\theta\,\partial_t + \left(1-\frac{GM}{r}\right)\cos\theta\,\partial_r - \frac{1}{r}\sin\theta\,\partial_\theta\,.
\end{aligned}}$$

Normalization $1 = -\xi\cdot\xi = \rho_1\cdot\rho_1 = \rho_2\cdot\rho_2 = \rho_3\cdot\rho_3$ and algebra $[\xi,\rho_I]_T = [\rho_I,\rho_J]_T = 0\,.$

Autoparallel Killing vectors

Sharif & Majeed (2009)
Petersen & Bonder (2019)
Boos (2025)

The autoparallel Killing equation $\nabla_\mu V_\nu + \nabla_\nu V_\mu = 0$ admits this maximal set of solutions:

$$\begin{aligned}
\xi &= \frac{r - GM}{r - 2GM}\,\partial_t + \frac{GM}{r}\,\partial_r\,, \\
\rho_1 &= \sin\theta\cos\varphi\left[\frac{GM}{r-2GM}\,\partial_t + \left(1 - \frac{GM}{r}\right)\,\partial_r\right] + \frac{1}{r}\cos\theta\cos\varphi\,\partial_\theta - \frac{1}{r}\frac{\sin\phi}{\sin\theta}\,\partial_\varphi\,, \\
\rho_2 &= \sin\theta\sin\varphi\left[\frac{GM}{r-2GM}\,\partial_t + \left(1 - \frac{GM}{r}\right)\,\partial_r\right] + \frac{1}{r}\cos\theta\sin\varphi\,\partial_\theta + \frac{1}{r}\frac{\cos\phi}{\sin\theta}\,\partial_\varphi\,, \\
\rho_3 &= \frac{GM}{r-2GM}\cos\theta\,\partial_t + \left(1 - \frac{GM}{r}\right)\cos\theta\,\partial_r - \frac{1}{r}\sin\theta\,\partial_\theta\,.
\end{aligned}$$

Normalization $1 = -\xi\cdot\xi = \rho_1\cdot\rho_1 = \rho_2\cdot\rho_2 = \rho_3\cdot\rho_3$ and algebra $[\xi, \rho_I]_T = [\rho_I, \rho_J]_T = 0\,.$

Vanishing mass: $\xi = \partial_t\,,\quad \rho_1 = \partial_x\,,\quad \rho_2 = \partial_y\,,\quad \rho_3 = \partial_z\,.$

Making sense of autoparallel Killing vectors

Boos (2025)

There is a map between $\{\xi, \rho_1, \rho_2, \rho_3\}$ and the Schwarzschild Killing vectors $\{\widetilde{\xi}, \widetilde{\rho}_1, \widetilde{\rho}_2, \widetilde{\rho}_3\}$:

$$0 = -\frac{GM}{r}\left(\rho_1 + \xi\cos\varphi\sin\theta\right) + \tilde{T}_{01} + \frac{1}{r}\tilde{T}_{23}$$

$$0 = -\frac{GM}{r}\left(\rho_2 + \xi\sin\varphi\sin\theta\right) + \tilde{T}_{02} + \frac{1}{r}\tilde{T}_{31}$$

$$0 = -\frac{GM}{r}\left(\rho_3 + \xi\cos\varphi\sin\theta\right) - \tilde{T}_{03} - \frac{1}{r}\tilde{T}_{12}$$

$$\tilde{T}_{0J} = \frac{1}{2}\epsilon_{\alpha\beta}{}^{\gamma\delta}T_{\alpha\beta}{}^{\mu}\widetilde{\xi}^{\gamma}\widetilde{\rho}^{\delta}_{J}\,\partial_\mu$$

$$\tilde{T}_{IJ} = \frac{1}{2}\epsilon_{\alpha\beta}{}^{\gamma\delta}T_{\alpha\beta}{}^{\mu}\widetilde{\rho}_{J}{}^{\gamma}\widetilde{\rho}^{\delta}_{J}\,\partial_\mu$$

More simply, one also has $\tilde{\rho}_I = \epsilon_I{}^{JK}T_{\alpha\beta}{}^{\mu}\rho^{\alpha}_{J}\rho^{\beta}_{J}$.

Interpretation: torsion transforms 3D translations into 3D rotations.

Makes intuitive sense, since a translation δ^K rotates around that axis with the generator $\epsilon_{IJK}\delta^K$.

4/5 Autoparallels

Complete set of autoparallel-conserved quantities

Boos (2025)

$$E = -g_{\mu\nu}\xi^\mu u^\nu \quad = \dot{t} - \frac{GM}{r}\left(\dot{t} + \frac{\dot{r}}{1-\frac{2GM}{r}}\right) \, ,$$

$$P_1 = +g_{\mu\nu}{\rho_1}^\mu u^\nu = \left[\frac{\mathrm{d}}{\mathrm{d}\tau} - \frac{GM}{r^2}\left(\dot{t} - \frac{\dot{r}}{1-\frac{2GM}{r}}\right)\right] r\sin\theta\cos\varphi \, ,$$

$$P_2 = +g_{\mu\nu}{\rho_2}^\mu u^\nu = \left[\frac{\mathrm{d}}{\mathrm{d}\tau} - \frac{GM}{r^2}\left(\dot{t} - \frac{\dot{r}}{1-\frac{2GM}{r}}\right)\right] r\sin\theta\sin\varphi \, ,$$

$$P_3 = +g_{\mu\nu}{\rho_3}^\mu u^\nu = \left[\frac{\mathrm{d}}{\mathrm{d}\tau} - \frac{GM}{r^2}\left(\dot{t} - \frac{\dot{r}}{1-\frac{2GM}{r}}\right)\right] r\cos\theta \, .$$

"Dispersion relation" $g_{\mu\nu}u^\mu u^\nu = -E^2 + P_1^2 + P_2^2 + P_3^2$.

Vanishing mass: $E = \dot{t}\,, \quad P_1 = \dot{x}\,, \quad P_2 = \dot{y}\,, \quad P_3 = \dot{z}\,.$

Autoparallels for equatorial motion

Boos (2025)

$$\dot{t} = \frac{(r - GM)E + GM\,(P_1 \cos\varphi + P_2 \sin\varphi)}{r - 2GM}\,,$$

$$\dot{r} = \left(1 - \frac{GM}{r}\right)(P_1 \cos\varphi + P_2 \sin\varphi) + \frac{GM}{r}E\,,$$

$$\dot{\varphi} = \frac{P_2 \cos\varphi - P_1 \sin\varphi}{r}\,.$$

$$\text{with} \quad -E^2 + P_1^2 + P_2^2 = \begin{cases} -1 & \text{timelike} \\ 0 & \text{null} \end{cases}$$

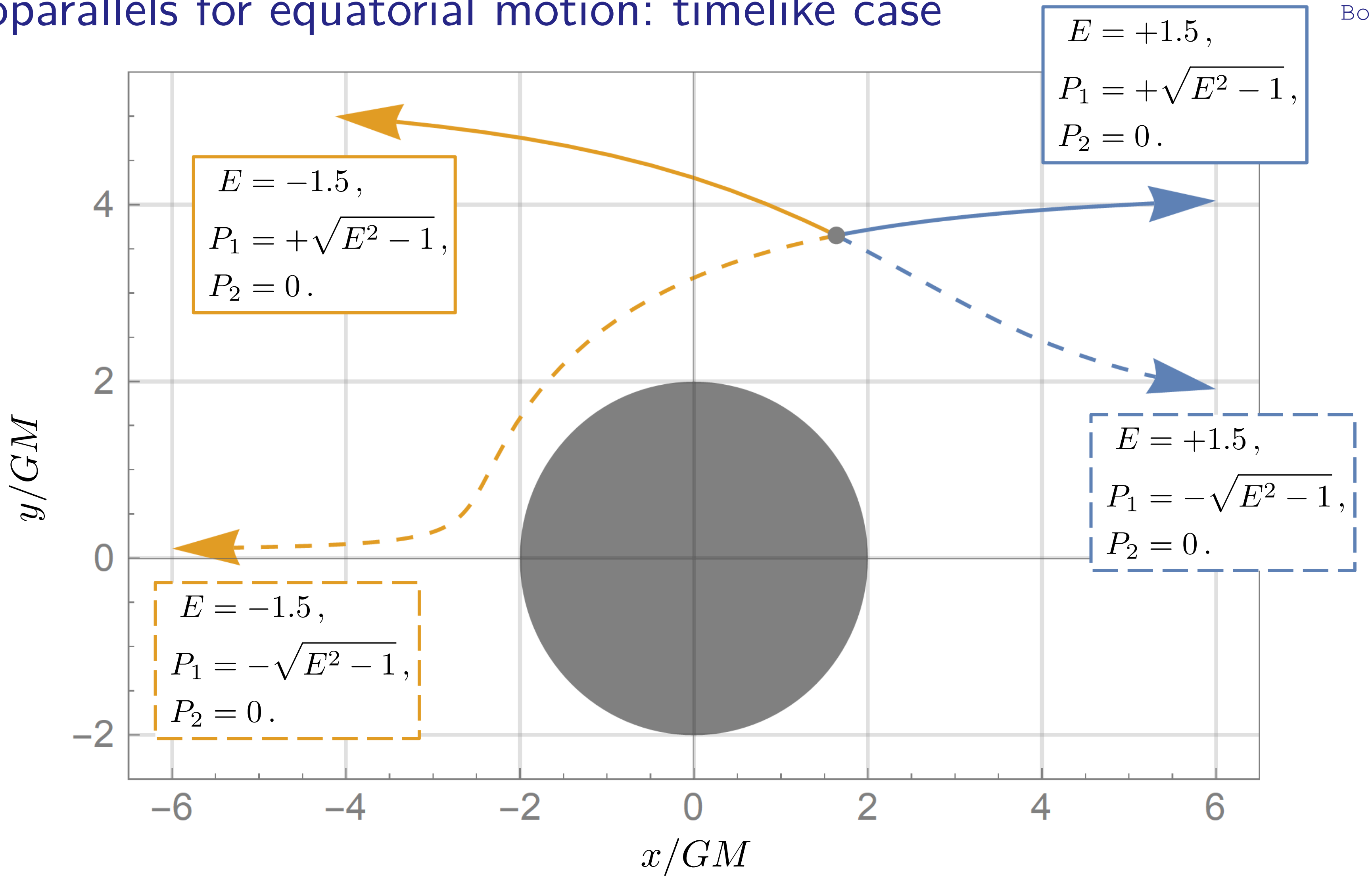
Equatorial motion is possible since $r\,\dot{\theta} = \cos\theta\,(P_1 \cos\varphi + P_2 \sin\varphi) + P_3 \sin\theta\,.$

But: Angular momentum is not conserved, $\tilde{L} = r\dot{\varphi} = P_2 \cos\varphi - P_1 \sin\varphi \neq \text{const.}$

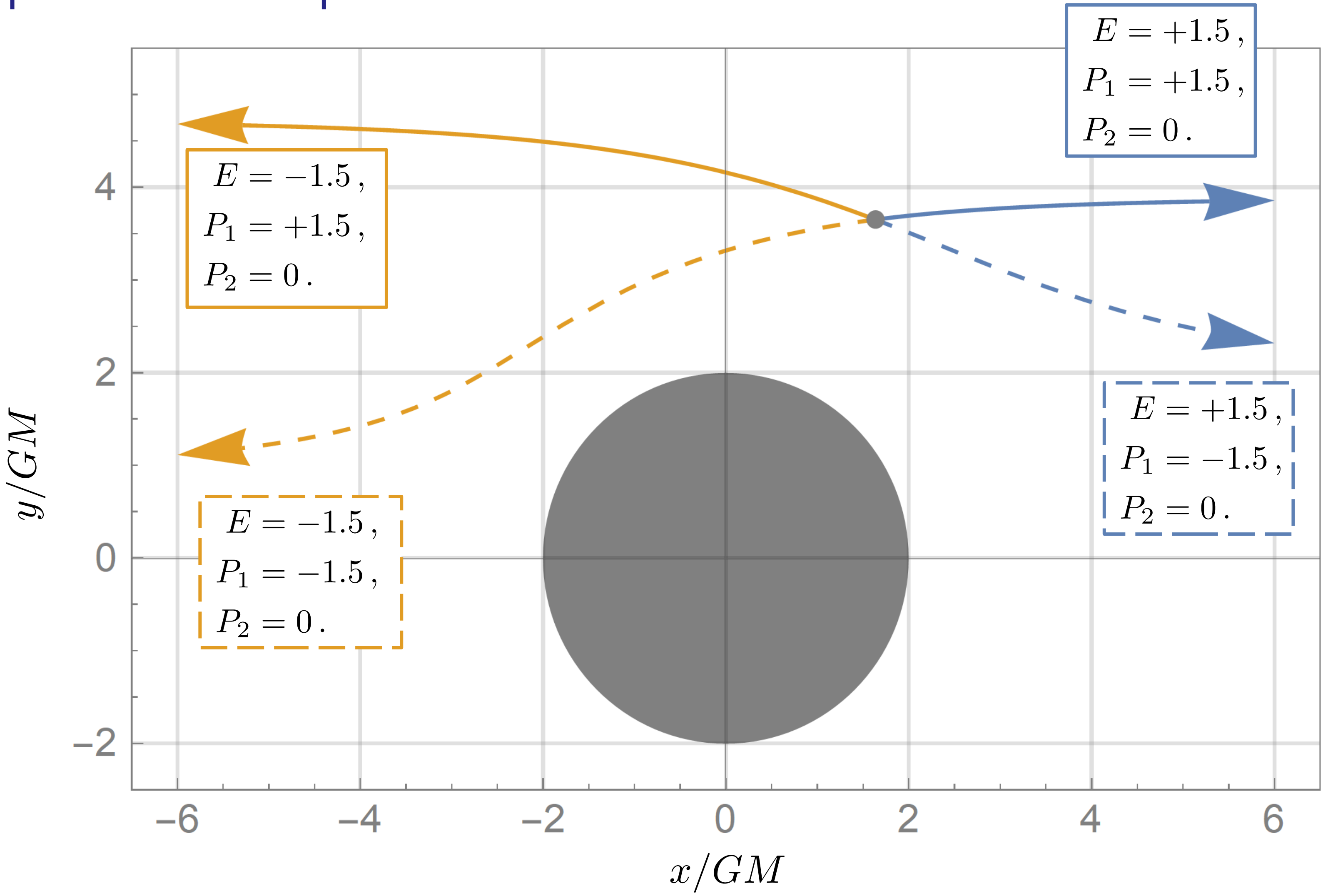
Only exception: $\tilde{L} = 0$

Autoparallels for equatorial motion: timelike case

Boos (2025)



Autoparallels for equatorial motion: null case



Radial infall

Setting $\dot{\varphi} = 0$ we obtain (dispersion relation eliminates another conserved quantity)

$$\dot{t} = \frac{(r - GM)E - GM\sqrt{E^2 - 1}}{r - 2GM}\,,$$

$$\dot{r} = \left(\frac{GM}{r} - 1\right)\sqrt{E^2 - 1} + \frac{GM}{r}E\,.$$

One may verify that $\dot{r} < 0$ for all r only if $E \leq -1$. Given $E > 1$, potential is attractive only for

$$r > r_\star(E) \equiv \left(1 + \sqrt{\frac{E^2}{E^2 - 1}}\right) GM\,.$$

Is this all just a sign error? Don't think so, switching ${T_{\mu\nu}}^\lambda \to -{T_{\mu\nu}}^\lambda$ gives non-constant curvature.

5/5 Summary

The good, the bad, the... autoparallel?

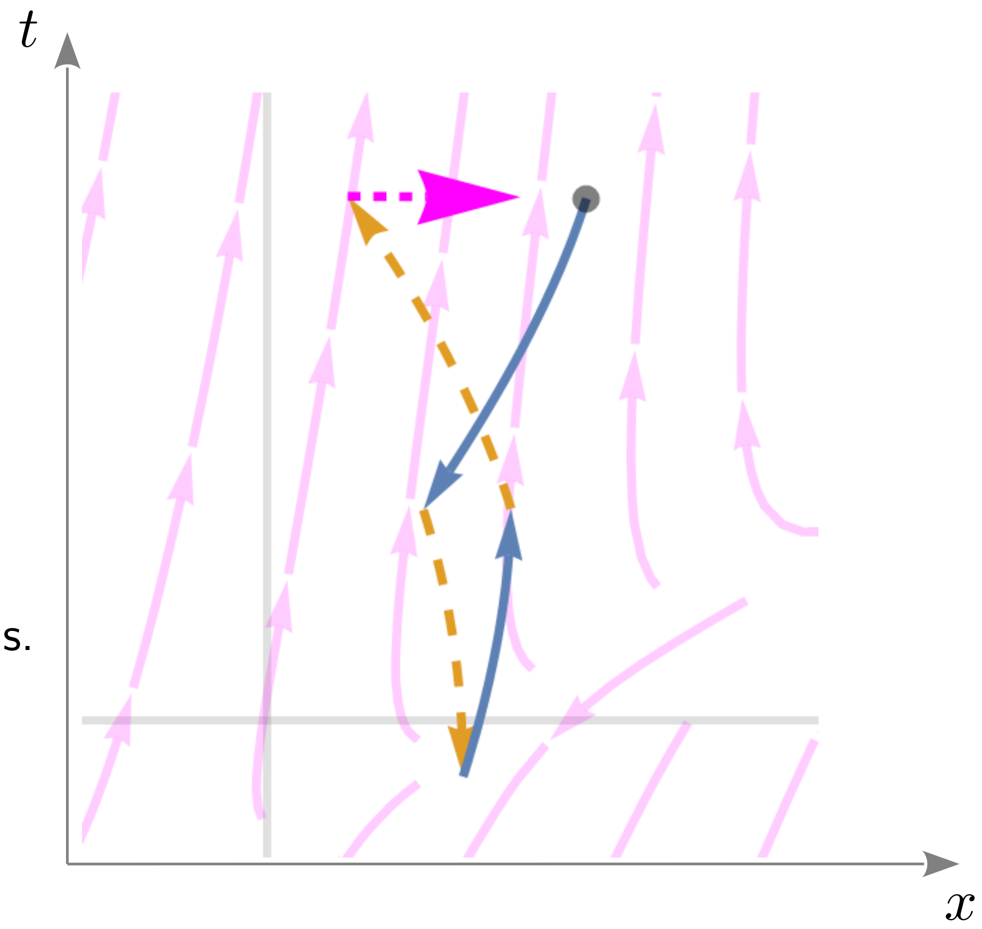
Mathematical insights:

- Four T-commuting autoparallel Killing vectors.
- Can be mapped to GR Killing vectors.
- Motion is integrable.

Physical status:

- Repulsive aspect needs to be understood.
- Obukhov + Puetzfeld: geodesics, not autoparallels.
- Mathisson–Papapetrou–Dixon equation?
- Effect less strong for "torsion hair" ?

Thank you for your attention.



Autoparallels around a Schwarzschild black hole with GM/r^2-torsion profile

We consider the autoparallel motion of test bodies around a Schwarzschild black hole endowed with a non-trivial torsion field scaling as GM/r^2, where M denotes the ADM mass of the black hole. By explicitly constructing a set of four orthogonal and commuting generalized Killing vectors and deriving their autoparallel conserved quantities we demonstrate the complete integrability of the equations of motion. Additionally, we study the qualitative orbital dynamics via effective potentials. Throughout, we compare the properties of the autoparallels (straightest possible paths) to that of the geodesics (shortest possible paths) and find notable discrepancies.