Quantum-mechanical scattering on a delta potential in ghost-free theory

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Quantum scattering on a delta potential in ghost-free theory

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Overview

1. What is a ghost-free theory?

2. Ghost-free research directions

3. Example: quantum-mechanical scattering
   a. Lippmann–Schwinger technique
   b. Green functions, analytic properties
   c. scattering coefficients

4. Outlook: quantum field theoretical vacuum expectation value

What is a ghost-free theory? (1/3)

Original motivation stems from Pauli–Villars regularization in quantum field theory. Let us give a simple example in the context of the classical theory of Newtonian gravity.

The gravitational potential of a point-particle is singular at the origin,

$$\nabla^2 \phi = 4\pi G m \delta(\boldsymbol{r}) \quad \longrightarrow \quad \phi(r) = \frac{-Gm}{r}$$

The pathological behavior at r=0 can be cured by introducing a heavy-mass modification:

$$\nabla^2 (1 - \nabla^2/M^2) \phi = 4\pi G m \delta(\boldsymbol{r}) \quad \longrightarrow \quad \phi(r) = \frac{-Gm}{r} \left(1 - e^{-Mr}\right)$$

This is called Pauli–Villars regularization, and we assume that $M \gg m$ (short distance modification). For large distances the potential is Newtonian, but for short distance scales it is regularized.

What is a ghost-free theory? (2/3)

The Green function of the Pauli–Villars regularized theory has the following structure:

$$G(\boldsymbol{x}, \boldsymbol{x'}) = \frac{1}{\nabla^2 \left(1 - \nabla^2/M^2\right)} = \frac{1}{\nabla^2} - \frac{1}{\nabla^2 - M^2}$$

The negative sign relative to the original propagator corresponds to a **ghost**. Using this Green function in quantum field theory can lead to negative probabilities and thereby violate unitarity.

What is the reason for this behavior? We have to blame mathematics, or, more precisely, the fundamental theorem of algebra: A non-constant polynomial of degree p has p complex roots.

Solution: Find a regularization that does not introduce any new zeroes!

What is a ghost-free theory? (3/3)

A very general solution is to modify the Poisson equation by a general function $f(\nabla^2/M^2)$:

$$\nabla^2 f(\nabla^2/M^2)\phi = 4\pi G m \delta(\boldsymbol{r})\,, \qquad G(\boldsymbol{x},\boldsymbol{x'}) = \frac{1}{\nabla^2 f(\nabla^2/M^2)}$$

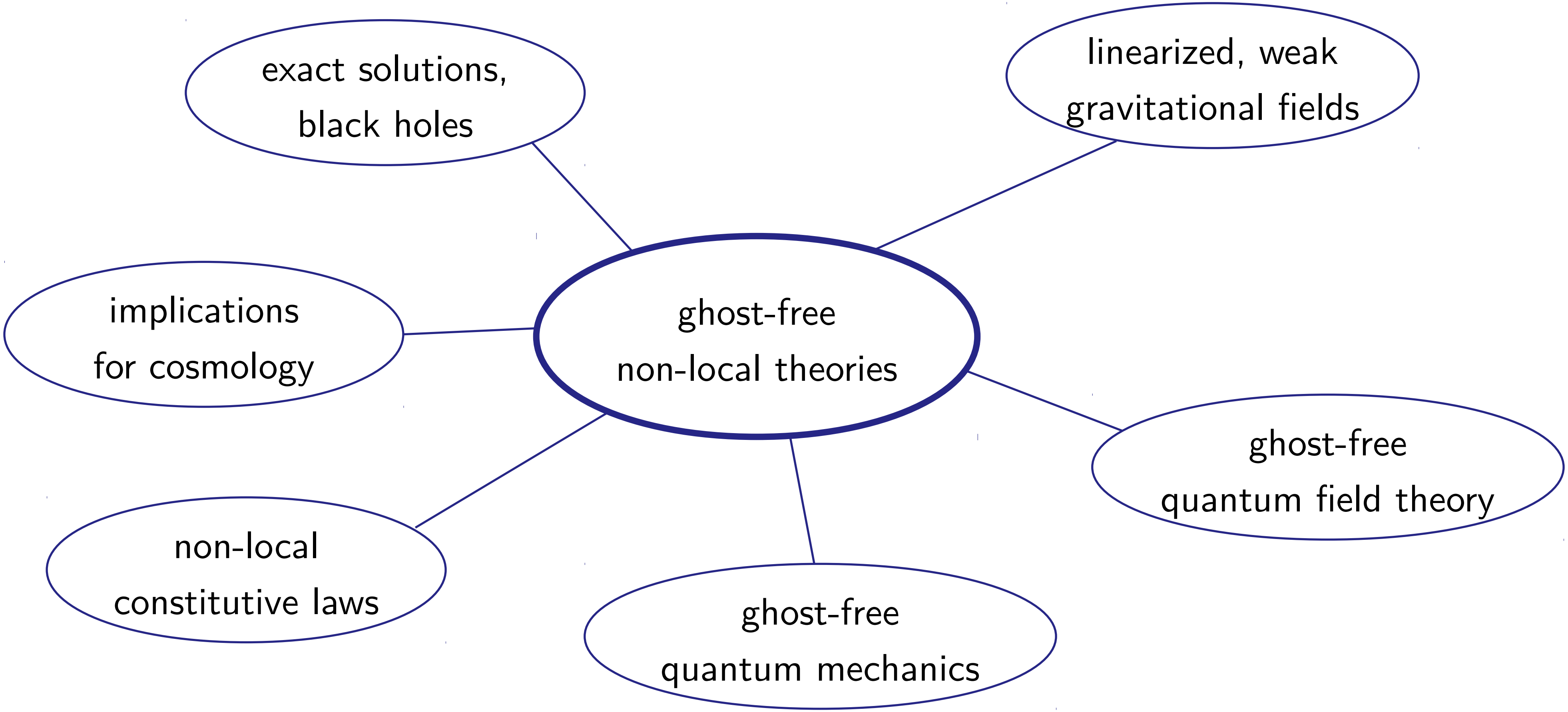
If f^{-1} is an entire function it has no poles in the complex plane and hence we can **avoid ghosts**. However, this comes at the cost of **locality:** performing a series expansion of $f(\nabla^2/M^2)$ introduces infinitely many derivative operators. For example, $e^{a\partial_x}h(x) = h(x+a)$ ➞ **non-locality.**

Intuitive explanation for the regularizing influence of the modification:

$$\nabla^2\phi = 4\pi G f^{-1}(\nabla^2/M^2) m\delta(\boldsymbol{r}) = 4\pi G \rho_{\text{eff}}(\boldsymbol{r})$$

Sharp delta-like distribution is smoothed out by the inverse operator $f^{-1}(\nabla^2/M^2)$.

What has been done? What can we study?



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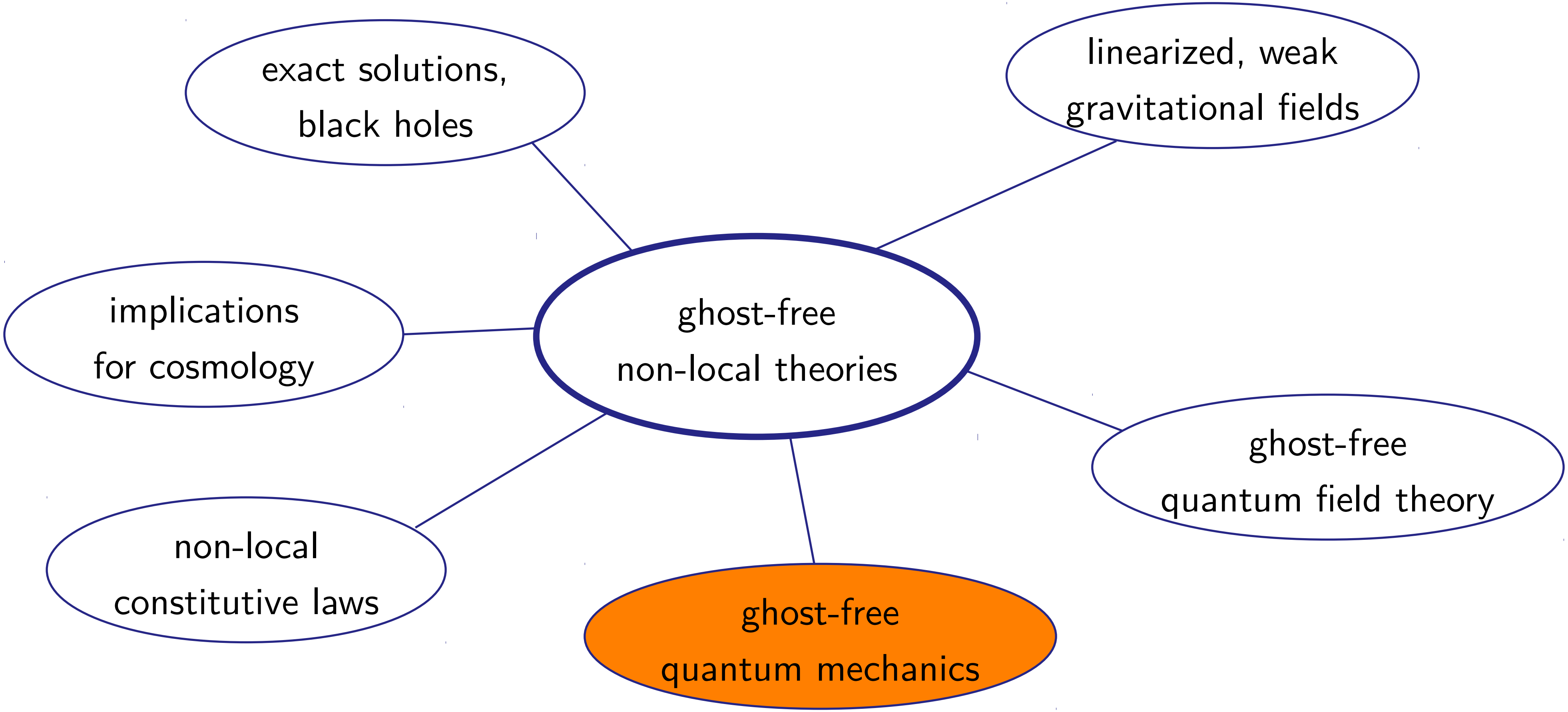


A simple problem: scattering of a scalar field on a delta-potential (1/2)

It is always good to study a simple example to gain some intuition. In this case, we were surprised by some interesting results. Consider the simple model of a scalar ghost-free theory in D=2:

$$S = \frac{1}{2}\int \mathrm{d}^2x \left[\varphi f(\Box\ell^2)\Box\varphi - V\varphi^2\right] , \quad f(\Box\ell^2) = \exp(-\ell^2\Box) , \quad V(x) = \lambda\delta(x) .$$

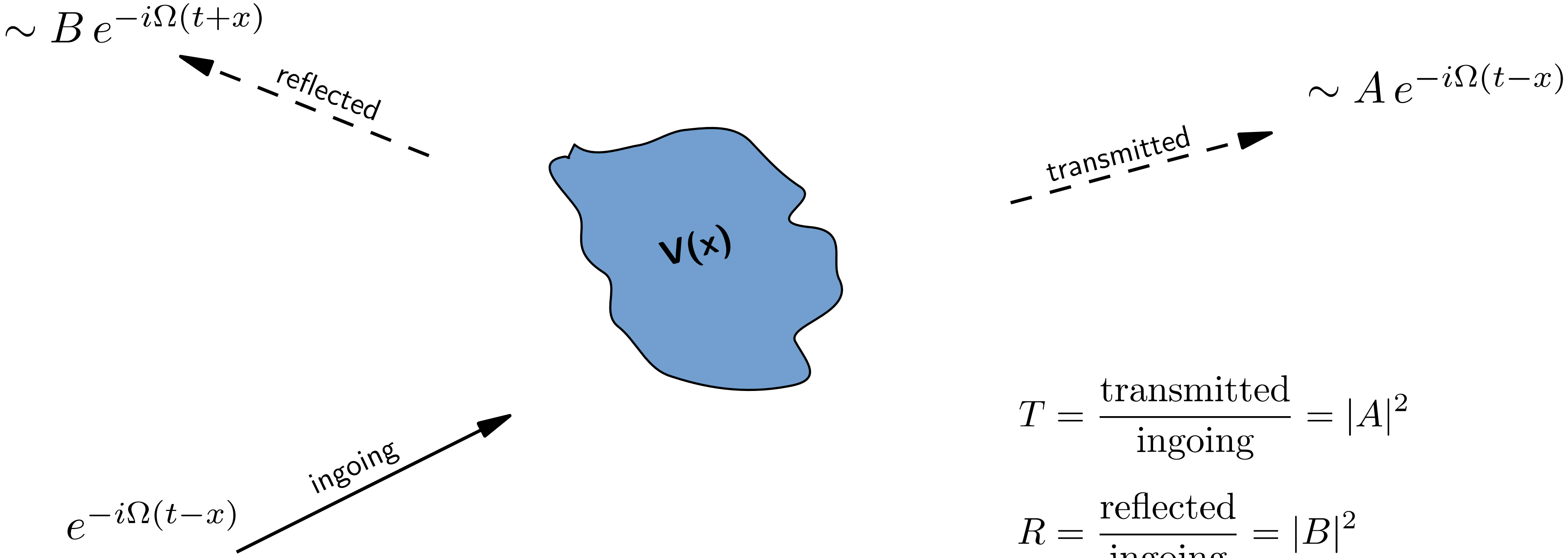
This **non-local modification** respects Poincaré invariance. The equation of motion is given by

$$\left\{\exp\left[-\ell^2(-\partial_t^2 + \partial_x^2)\right](-\partial_t^2 + \partial_x^2) - \lambda\delta(x)\right\}\varphi(t,x) = 0 .$$

You might wonder: doesn't an infinite-order equation of motion require an infinite amount of initial conditions? The answer: each pole of the propagator contributes two pieces of initial data to the final solution (Barbaby & Kamran, JHEP 2008).

A simple problem: scattering of a scalar field on a delta-potential (2/2)

The scattering coefficients can be read off by the asymptotic behavior of the field:



$$T = \frac{\text{transmitted}}{\text{ingoing}} = |A|^2$$

$$R = \frac{\text{reflected}}{\text{ingoing}} = |B|^2$$

The Lippmann–Schwinger approach to the scattering problem

Given a differential operator $\mathcal{D}_x$ one can determine associated Green functions $G(\boldsymbol{x}, \boldsymbol{x'})$ by solving $\mathcal{D}_x G(\boldsymbol{x}, \boldsymbol{x'}) = -\delta(\boldsymbol{x} - \boldsymbol{x'})$. Let us focus on the **scattering problem**.

$$[\mathcal{D}_x - V(\boldsymbol{x})]\,\varphi(\boldsymbol{x}) = 0$$

is solved by

$$\boxed{\varphi(\boldsymbol{x}) = \varphi_0(\boldsymbol{x}) - \int \mathrm{d}\boldsymbol{x'} G_0^{\mathrm{R}}(\boldsymbol{x} - \boldsymbol{x'}) V(\boldsymbol{x'}) \varphi(\boldsymbol{x'})}$$

It is important to use the correct Green function. Here, in our example of a quantum-mechanical scattering problem, we will employ the free retarded Green function $G_0^{\mathrm{R}}(\boldsymbol{x} - \boldsymbol{x'})$.

The choice of Green functions and their analytic properties

Analytic properties in the momentum space representation fix the Green function uniquely:

$$G(\boldsymbol{x},\boldsymbol{x'}) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\Omega}{2\pi} e^{-i\Omega(t-t')} \widetilde{G}_\Omega(x,x') = \int_{-\infty}^{\infty} \frac{\mathrm{d}\Omega}{2\pi} e^{-i\Omega(t-t')} \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} e^{+ik(x-x')} \times \begin{cases} \boxed{-\dfrac{1}{(\Omega - i\epsilon)^2 - k^2} \quad \text{retarded}} \\ -\dfrac{1}{(\Omega + i\epsilon)^2 - k^2} \quad \text{advanced} \\ -\dfrac{1}{\Omega^2 - k^2 + i\epsilon} \quad \text{Feynman} \end{cases}$$

The above relations are for the local case where $\ell = 0$. We can prove that it remains valid for $\ell \neq 0$. This is important, since in the non-local case it is difficult to define a 'local retardation' and so on.

Non-local ghost-free Green functions (1/3)

We define the **non-local** equivalents of these Green functions as follows:

$$G(\boldsymbol{x}, \boldsymbol{x}') = \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\Omega}{2\pi} e^{-i\Omega(t-t')} \widetilde{G}_\Omega(x, x')$$

$$= \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\Omega}{2\pi} e^{-i\Omega(t-t')} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} e^{+ik(x-x')} \times \begin{cases} -\dfrac{e^{-\ell^2(k^2-\Omega^2)}}{(\Omega - i\epsilon)^2 - k^2} & \text{retarded} \\ -\dfrac{e^{-\ell^2(k^2-\Omega^2)}}{(\Omega + i\epsilon)^2 - k^2} & \text{advanced} \\ -\dfrac{e^{-\ell^2(k^2-\Omega^2)}}{\Omega^2 - k^2 + i\epsilon} & \text{Feynman} \end{cases}$$

Why is this consistent? It looks like the exponential term quite heavily modifies the analytic behavior in the k-plane. In particular, there seems to be a divergence for large imaginary k.

Non-local ghost-free Green functions (2/3)

Let us write the total Green function as a local one plus a modification term:

$$\widetilde{G}_\Omega(x - x') = \widetilde{G}_\Omega^{\mathrm{loc}}(x - x') + \Delta\widetilde{G}_\Omega(x - x')$$

We can rewrite the modification term and thereby prove that the non-local contribution is finite:

$$\begin{aligned}\Delta\widetilde{G}_\Omega(x - x') &= \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} e^{+ik(x-x')} \frac{e^{-\ell^2(k^2-\Omega^2)} - 1}{k^2 - \Omega^2} \\ &= -\int\limits_0^{\ell^2} \mathrm{d}s \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} e^{+ik(x-x')} e^{-s(k^2-\Omega^2)}\end{aligned}$$

This integral is convergent and can be solved analytically.

Non-local ghost-free Green functions (3/3)

Employing the retarded $i\epsilon$-prescription ($\Omega \to \Omega - i\epsilon$) we obtain for the integral

$$\widetilde{G}^{\mathrm{R}}_{\Omega}(x,x') = \frac{i}{4\Omega}\left\{ \quad e^{+i\Omega(x-x')}\left[1+\operatorname{erf}\left(i\Omega\ell + \frac{x-x'}{2\ell}\right)\right] \right.$$
$$\left. + e^{-i\Omega(x-x')}\left[1+\operatorname{erf}\left(i\Omega\ell - \frac{x-x'}{2\ell}\right)\right]\right\} \xrightarrow{\ell \to 0} \frac{i}{2\Omega}e^{i\Omega|x-x'|}$$

In the limit of vanishing non-locality one recovers the local propagator. This is automatically enforced by the condition that f(0) $=$ 1 in ghost-free theories.

Note: This has interesting implications for on-shell vs. off-shell quantities. Whenever the expression $f(\Box\ell^2)$ acts on an object X that satisfies the field equations, one has $f(\Box\ell^2)X = f(0)X = X$. This makes it difficult to find effects of non-locality in many scenarios.

Fourier decomposition

The real field $\varphi(\boldsymbol{x})$ admits a Fourier decomposition into complex modes $\widetilde{\varphi}_\Omega(x)$ which solve

$$\Big\{ \exp\left[-\ell^2(\Omega^2 + \partial_x^2)\right](\Omega^2 + \partial_x^2) - \lambda\delta(x)\Big\}\widetilde{\varphi}_\Omega(x) = 0\,.$$

We can apply the Lippmann-Schwinger technique to the Fourier modes directly. As a free solution, let us choose a wave $\widetilde{\varphi}_{0,\Omega}(x) = e^{i\Omega x}$. Then, the exact solution can be written as

$$\widetilde{\varphi}_\Omega(x) = e^{i\Omega x} - \frac{\lambda}{1 + \lambda \widetilde{G}^{\mathrm{R}}_\Omega(0)} \widetilde{G}^{\mathrm{R}}_\Omega(x)\,,$$

provided that $1 + \lambda \widetilde{G}^{\mathrm{R}}_\Omega(0) \neq 0$ (this condition selects scattering states and excludes bound states). Identifying in- and out-going modes, we can now read off the scattering coefficients.

Result: non-local scattering coefficients (1/3)

Read off asymptotically **in-going** and **out-going** modes:

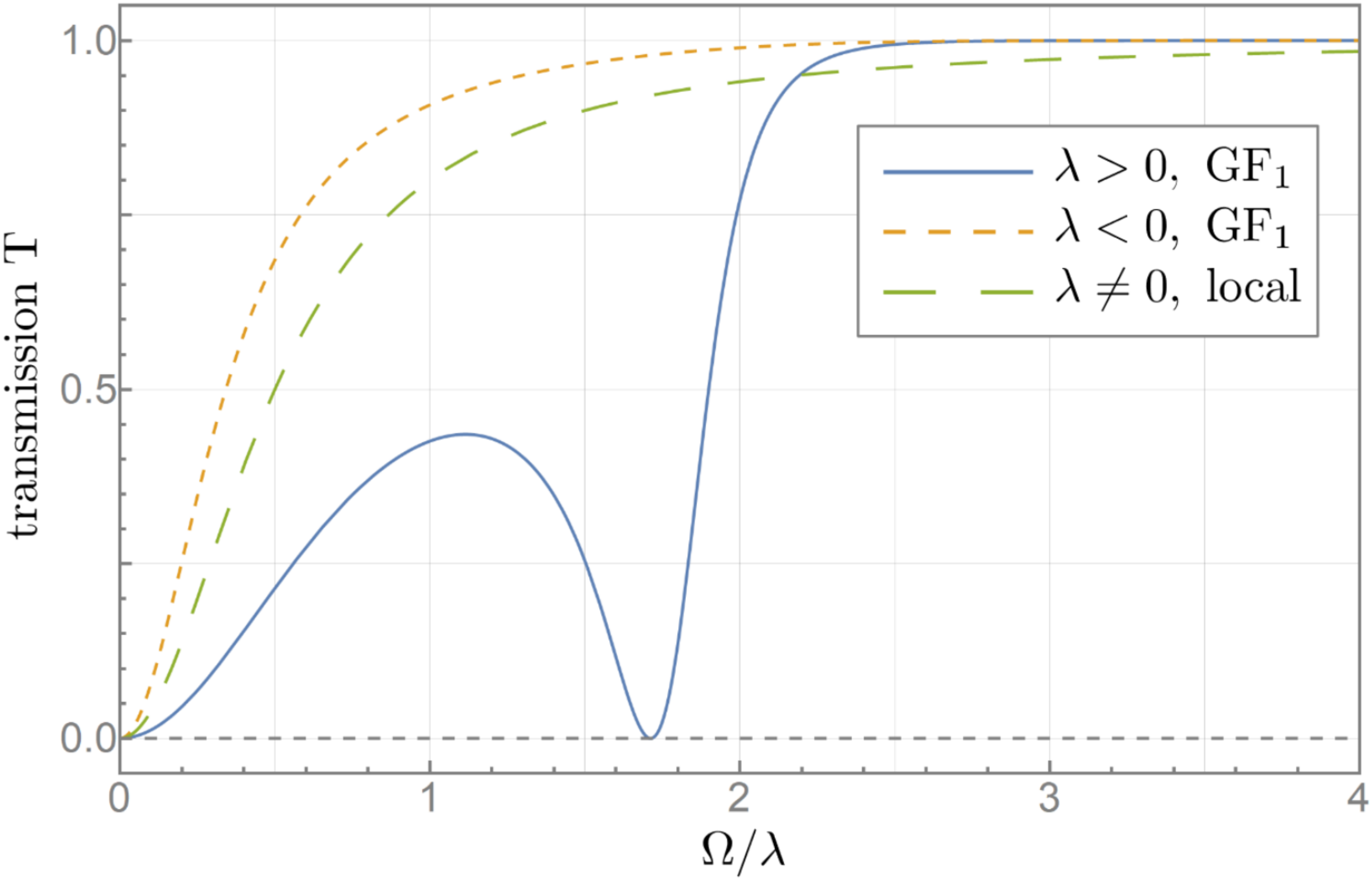
$$\widetilde{\varphi}_\Omega(x) = e^{i\Omega x} - \frac{\lambda}{1+\lambda \widetilde{G}^{\mathrm{R}}_\Omega(0)} \frac{i}{4\Omega} \left\{ e^{+i\Omega x}\left[1+\mathrm{erf}\left(i\Omega\ell + \frac{x}{2\ell}\right)\right] + e^{-i\Omega x}\left[1+\mathrm{erf}\left(i\Omega\ell - \frac{x}{2\ell}\right)\right] \right\}$$

Explicitly, the transmission coefficient T has the form (reflection $R = 1 - T$):

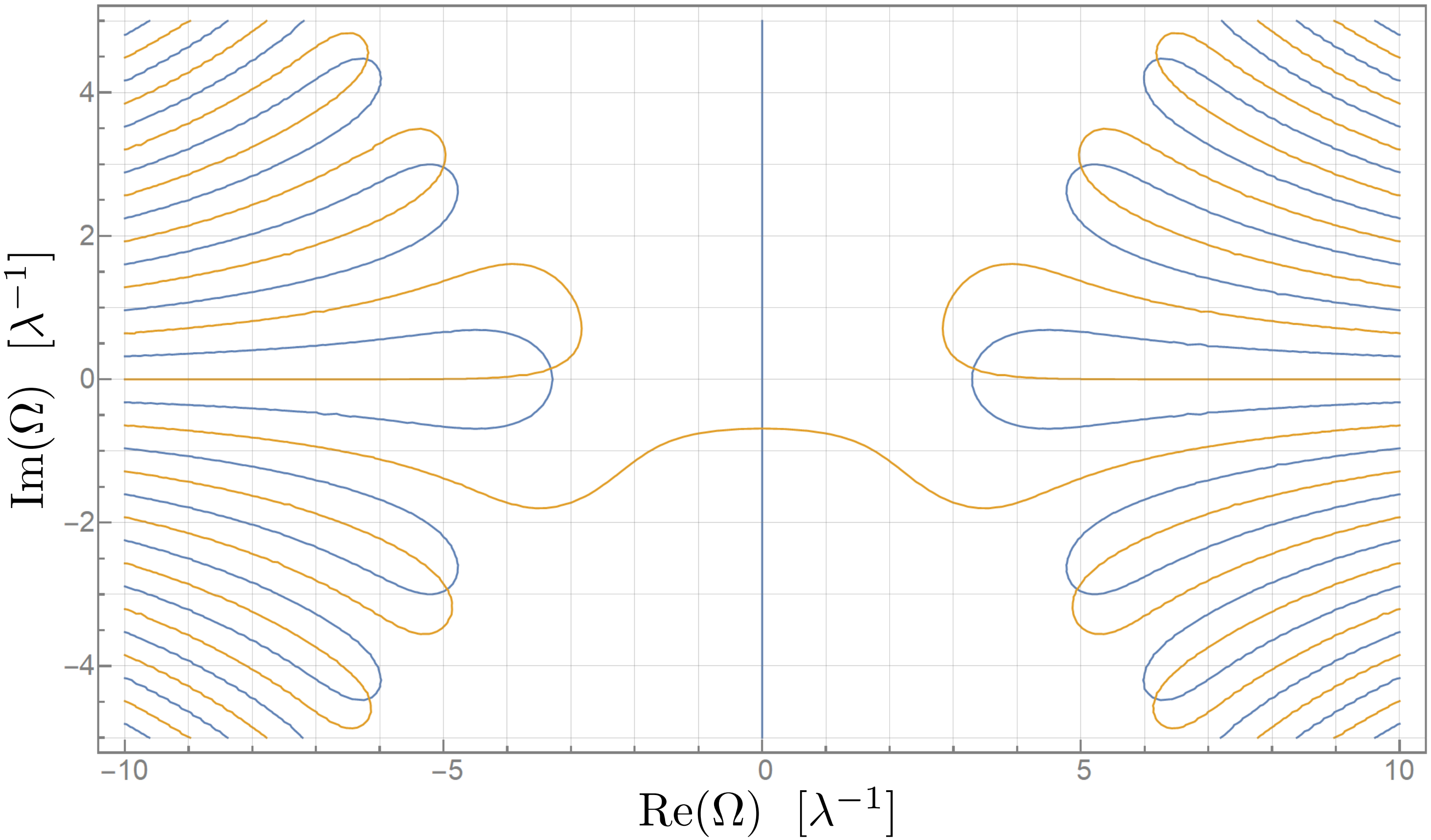
$$T = \frac{4\gamma^2}{1+4\gamma^2}, \quad \gamma = \frac{\Omega}{\lambda} - \frac{1}{2}\mathrm{erfi}\,(\Omega\ell) \longrightarrow \begin{cases} 1 \text{ for } \Omega\ell \gg 1 \text{ even if } \lambda \to \infty \\ 0 \text{ for } \Omega = \Omega_\star \text{ provided } \lambda\ell < \sqrt{\pi} \end{cases}$$

These are unexpected results. We are open to physical interpretations!

Result: non-local scattering coefficients (2/3)

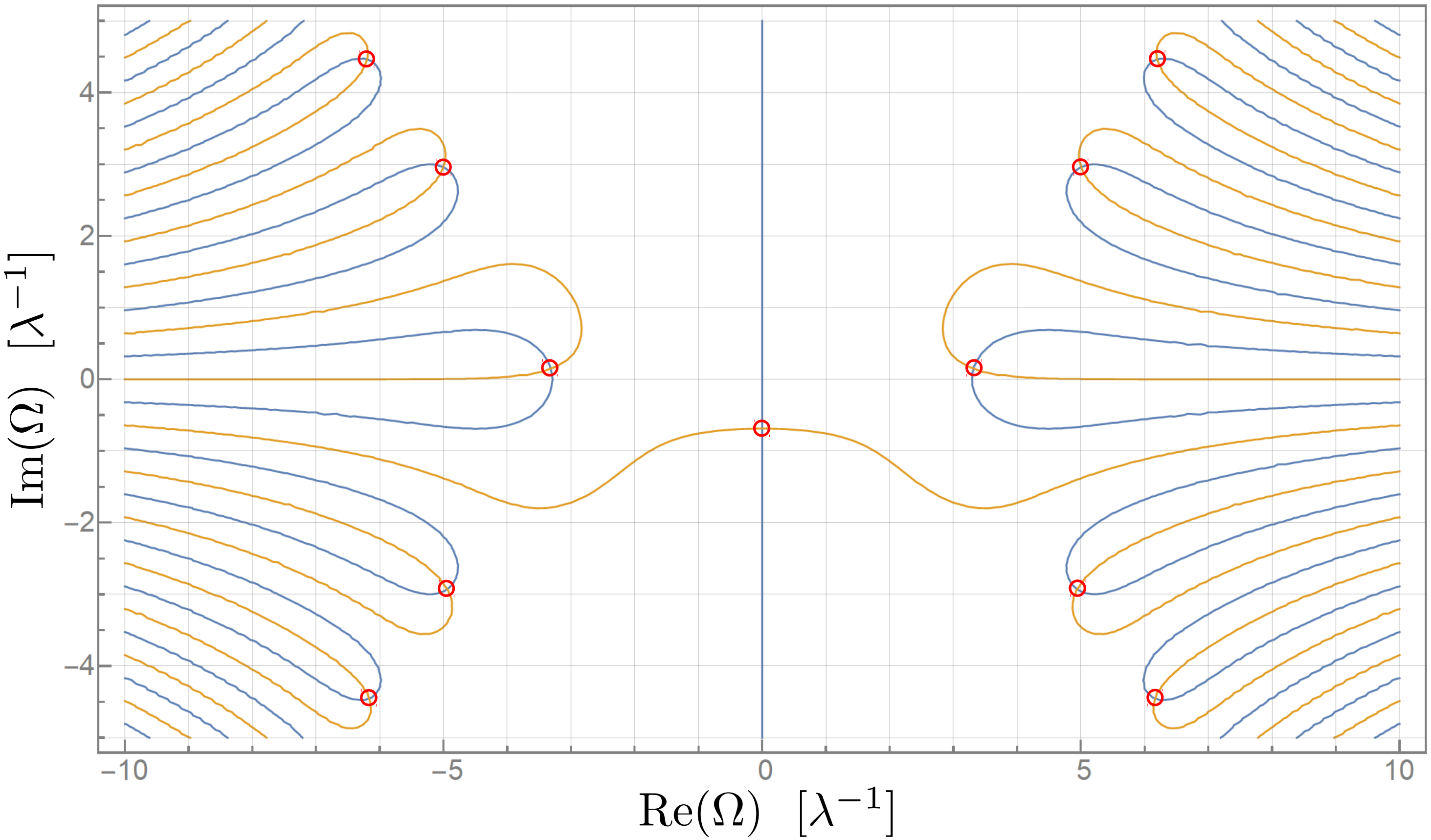


Result: quasi-normal modes (3/3)



Numerical plot of the **real** and **imaginary** zeroes of $2\Omega/\lambda + i - \mathrm{erfi}\,(\Omega\ell) = 0$ for $\lambda\ell = 0.5$.

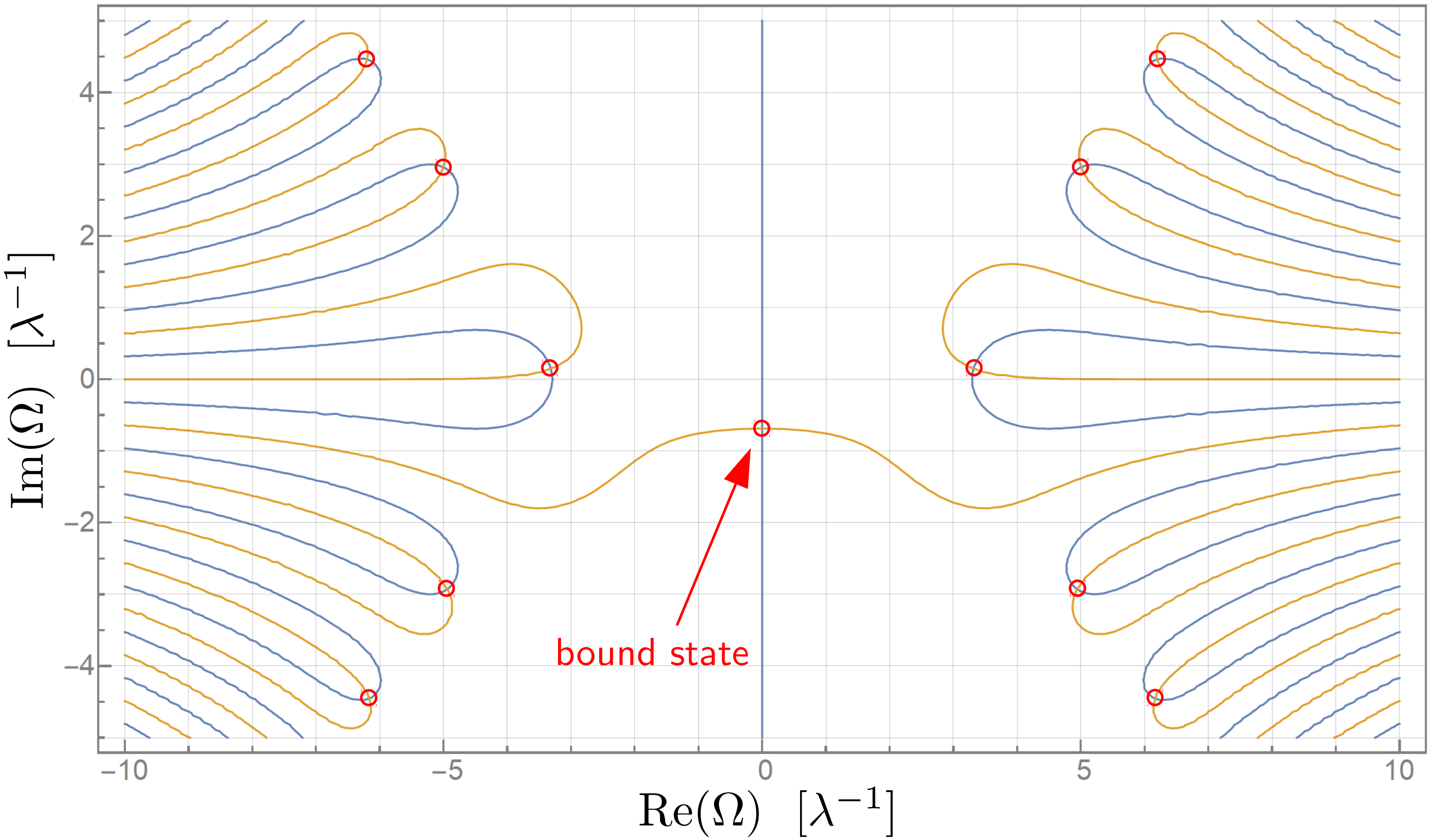
Result: quasi-normal modes (3/3)



Numerical plot of the **real** and **imaginary** zeroes of $2\Omega/\lambda + i - \mathrm{erfi}\,(\Omega\ell) = 0$ for $\lambda\ell = 0.5$.

Every **intersection** of the blue and orange line corresponds to one complex root.

Result: quasi-normal modes (3/3)



Numerical plot of the **real** and **imaginary** zeroes of $2\Omega/\lambda + i - \operatorname{erfi}(\Omega\ell) = 0$ for $\lambda\ell = 0.5$.

Every **intersection** of the blue and orange line corresponds to one complex root.

Conclusions and outlook

The field of ghost-free physics is very interesting and has many open problems.
Instead of focusing on conceptual issues, we find it insightful to study concrete problems, such as

- the weak-field limit of ghost-free gravity
- quantum-mechanical scattering problem
- vacuum polarization in quantum field theory
- the Casimir effect

It would be interesting to extend these studies into the realm of curved spacetimes to understand the notion of non-locality close to a BH horizon.

Thank you for your attention.

Abstract

We discuss the quantum-mechanical scattering of a massless scalar field on a δ-potential in a ghost-free theory and obtain analytic solutions for the scattering coefficients. There are a few interesting results:

- Due to the non-locality of the ghost-free theory the transmission coefficient tends to unity for frequencies much larger than the inverse scale of non-locality, even for infinitely strong potentials.
- At the same time there exists a critical strength of the δ-potential barrier below which there is always a frequency that is totally reflected.

These scattering properties in ghost-free theories are quite generic and distinguish them from local field theories. Moreover, we study quasi-normal states that are present for the δ-potential well. In the limit of vanishing non-locality, we recover the standard results of local field theory.
If time permits, we will outline our current efforts in understanding the structure of the ghost-free vacuum in the context of quantum field theory: this includes modifications to the vacuum fluctuations of a scalar field, as well as possible modifications of the Casimir effect.