An exact Kerr-(A)dS black hole solution with torsion and curvature

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I would like to thank my teacher **Friedrich W. Hehl** for many insightful discussions.

Gravity seminar
University of Alberta, Edmonton
Friday, April 12, 2019, 12:30pm, CCIS 4-285

It is a good week to talk about the Kerr metric...

April 10, 2019, Event Horizon Telescope Collaboration

My work on Poincaré gauge gravity with Friedrich W. Hehl

1. Premetric teleparallel theory of gravity and its local and linear constitutive law, Eur. Phys. J. C **78** (2018) no. 11; 1808.08048 [gr-qc] (with FWH, Yakov Itin, Yuri Obukhov)

2. Gravity-induced four-fermion contact interaction implies gravitational intermediate W and Z type gauge bosons, Int. J. Theor. Phys. **56** (2017) no. 3; 1606.09273 [gr-qc].

3. Quasi-normal modes of the BTZ black hole solution of (2+1)-dimensional topological Poincare gauge gravity, Master's thesis, University of Cologne, 2015.

Overview

1. Motivation to look beyond Einstein's Riemannian manifolds?
2. Brief history of Poincaré gauge gravity

3. Riemann–Cartan geometry
4. Dynamical framework of Poincaré gauge gravity

5. Kerr-(A)dS solution of General Relativity
6. Kerr-(A)dS solution of Poincaré gauge gravity

7. Summary & open problems

PART I

INTRODUCTION

Why look into geometries beyond Einstein's Riemannian manifolds?

Physical reasons:

- equivalence principle valid for fermions, not just dust particles ("Kibble's laboratory")
- teleparallel equivalent of General Relativity (gravity=force as opposed to gravity=geometry)
- can understand gravity as a gauge theory of the Poincaré group
- torsion plays an eminent role in supergravity

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Mathematical reasons:

- Which properties/structures/exact solutions are germane to Riemannian manifolds?
- What carries over to more general settings?
- classification of spacetimes is interesting

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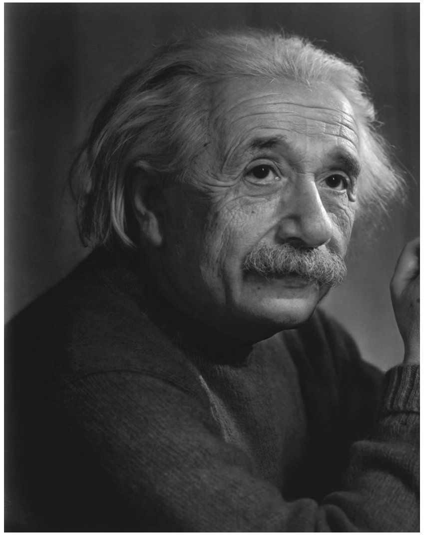
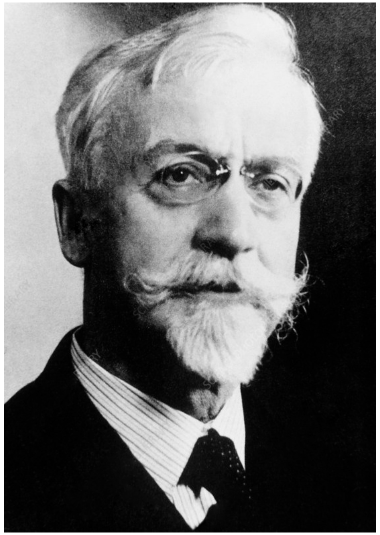
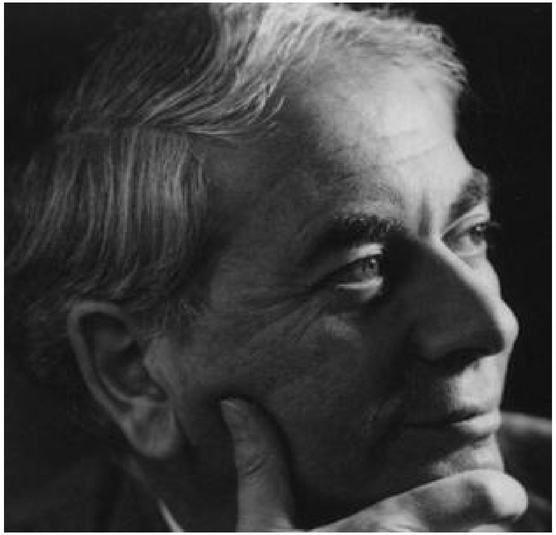
Pragmatic reason: all solutions of GR are included in Poincaré gauge gravity.

Brief history of Poincaré gauge gravity


Albert Einstein (1879–1955)


Élie Cartan (1869–1951)


Dennis Sciama (1926–1999)


Tom W. Kibble (1932–2016)

After Einstein's General Relativity, Cartan developed the theory of affine connections (Cartan '23).

Much later, Sciama ('60) and Kibble ('61) independently proposed a gauge theory of gravity based on the Poincaré group, in complete analogy to the then-recent Yang–Mills theory from 1954.

This development was continued by Hehl, Hayashi, Trautman, and many others in the 1970s.

Later: metric-affine gravity based on conformal group (Hehl, McCrea, Mielke, Ne'eman '95)

PART II

POINCARÉ GAUGE GRAVITY

Spacetimes with curvature and torsion = Riemann–Cartan geometry

Around an infinitesimal loop a scalar function f and a vector V^k pick up the following holonomy:

$$[\nabla_i, \nabla_j] f = -T_{ij}{}^a \nabla_a f \,, \quad [\nabla_i, \nabla_j] V^k = R_{ij}{}^k{}_a V^a - T_{ij}{}^a \nabla_a V^k \,.$$

Here, $T_{ij}{}^k$ is the spacetime torsion, and $R_{ij}{}^k{}_l$ is the spacetime curvature.

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Here, $T_{ij}{}^k$ is the spacetime torsion, and $R_{ij}{}^k{}_l$ is the spacetime curvature. Given a tetrad $e_i{}^\mu$ as well as a metric-compatible Lorentz connection $\Gamma_i{}^\mu{}_\nu$, we can write torsion and curvature as

$$\begin{aligned} T_{ij}{}^\mu &= 2\partial_{[i} e_{j]}{}^\mu + 2\Gamma_{[i}{}^\mu{}_\alpha e_{j]}{}^\alpha \,, \\ R_{ij}{}^\mu{}_\nu &= 2\partial_{[i} \Gamma_{j]}{}^\mu{}_\nu + 2\Gamma_{[i}{}^\mu{}_\alpha \Gamma_{j]}{}^\alpha{}_\nu \,. \end{aligned}$$

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Fundamental "potential" 1-forms: coframe $\vartheta^{\mu} = e_a{}^{\mu} \mathrm{d}x^a$ and Lorentz connection $\Gamma^{\mu}{}_{\nu} = \Gamma_a{}^{\mu}{}_{\nu} \mathrm{d}x^a$.

Dynamical framework of Poincaré gauge gravity

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translational and rotational excitations

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gravitational energy-momentum and spin-angular momentum

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compare to classical electrodynamics

$$\mathscr{L} = \tfrac{1}{2} F \wedge \star F + \mathscr{L}_{\mathrm{m}}$$

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PART III

EXACT SOLUTIONS

Kerr–NUT–(A)dS solution of General Relativity

Orthonormal coframe in Carter's canonical coordinates ('68):

$$\vartheta^{\hat{0}} = \sqrt{\frac{\Delta_r}{\Sigma}}\left(\mathrm{d}\tau + y^2\mathrm{d}\psi\right)\,,\ \vartheta^{\hat{1}} = \sqrt{\frac{\Sigma}{\Delta_r}}\mathrm{d}r\,,\ \vartheta^{\hat{2}} = \sqrt{\frac{\Sigma}{\Delta_y}}\mathrm{d}y\,,\ \vartheta^{\hat{3}} = \sqrt{\frac{\Delta_y}{\Sigma}}\left(\mathrm{d}\tau - r^2\mathrm{d}\psi\right)\,,$$

$$\boldsymbol{g} = -\vartheta^{\hat{0}}\otimes\vartheta^{\hat{0}} + \vartheta^{\hat{1}}\otimes\vartheta^{\hat{1}} + \vartheta^{\hat{2}}\otimes\vartheta^{\hat{2}} + \vartheta^{\hat{3}}\otimes\vartheta^{\hat{3}}\,, \qquad \Sigma = r^2 + y^2\,.$$

Auxiliary functions:

$$\Delta_r = (r^2 + a^2)(1 - \Lambda r^2/3) - 2Mr\,, \quad \Delta_y = (a^2 - y^2)(1 + \Lambda y^2/3) + 2Ny\,.$$

This metric solves the vacuum Einstein equations with cosmological constant Λ.

- M = mass of the black hole
- a = angular momentum parameter, $J = Ma$
- N = NUT parameter

Kerr–NUT–(A)dS solution of Poincaré gauge gravity?

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For now, let us focus on the Kerr–(A)dS solution instead. This goes back to the following works:

- Baekler ('81); Benn, Dereli, and Tucker (‘81); Lee (‘83)
- McCrea ('84); Baekler and Hehl (‘84)
- Baekler, McCrea, and Guerses ('87)
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The “Golden Ages” of Poincaré gauge gravity: solution-generating methods meet computer algebra!

Kerr–(A)dS solution of Poincaré gauge gravity

The Lagrangian is quadratic in torsion and curvature:

$$\mathscr{L}_{\mathrm{g}} = -\frac{1}{2\kappa}\left(T^{\alpha} \wedge \vartheta^{\beta}\right) \wedge \star\left(T_{\beta} \wedge \vartheta_{\alpha}\right) - \frac{1}{2\rho} R_{\alpha\beta} \wedge \star R^{\alpha\beta}$$

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Two coupling constants:

- Einstein's gravitational constant: $\kappa = \frac{8\pi G}{c^4}$
- a new "strong gravity"-type dimensionless constant: ρ

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Remarks:

- no linear pieces in curvature, no cosmological constant term
- Lagrangian can be regarded as a gravitational analogue of the Yang–Mills Lagrangian
- the torsion-square piece alone gives rise to the standard Newtonian limit

Kerr–(A)dS solution of Poincaré gauge gravity

Orthonormal coframe:

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This metric solves the vacuum Einstein equations with effective cosmological constant $\Lambda_{\mathrm{eff}} = \dfrac{3\rho}{4\kappa}$.

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Kerr–(A)dS solution of Poincaré gauge gravity

The torsion barely fits on one screen:

$$T^{\hat{0}} = T^{\hat{1}} = \sqrt{\frac{\Sigma}{\Delta_r}} \left(-v_1 \vartheta^{\hat{0}} \wedge \vartheta^{\hat{1}} - 2v_4 \vartheta^{\hat{2}} \wedge \vartheta^{\hat{3}} \right) + \frac{\Sigma}{\Delta_r} \left(\vartheta^{\hat{0}} - \vartheta^{\hat{1}} \right) \wedge \left(v_2 \vartheta^{\hat{2}} + v_3 \vartheta^{\hat{3}} \right),$$

$$T^{\hat{2}} = \sqrt{\frac{\Sigma}{\Delta_r}} \left(\vartheta^{\hat{0}} - \vartheta^{\hat{1}} \right) \wedge \left(+v_5 \vartheta^{\hat{2}} + v_4 \vartheta^{\hat{3}} \right),$$

$$T^{\hat{3}} = \sqrt{\frac{\Sigma}{\Delta_r}} \left(\vartheta^{\hat{0}} - \vartheta^{\hat{1}} \right) \wedge \left(-v_4 \vartheta^{\hat{2}} + v_5 \vartheta^{\hat{3}} \right),$$

$$v_1 = \frac{m(r^2 - y^2)}{\Sigma^4}, \quad v_2 = -\frac{mr\sqrt{\Delta_y}\, y \sqrt{a^2 - y^2}}{\Sigma^5}, \quad v_3 = \frac{mr^2 \sqrt{\Delta_y} \sqrt{a^2 - y^2}}{\rho^5},$$

$$v_4 = \frac{mry}{\Sigma^4}, \quad v_5 = \frac{mr^2}{\Sigma^4}.$$

Properties: proportional to mass, null, vanishes at spatial infinity, and $T_{[\mu\nu\rho]} = 0$.

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What about the curvature of the solution?

$$R = 4\Lambda_{\text{eff}} = \frac{3\rho}{\kappa} = \text{const}$$

Can also calculate purely Riemannian curvature based on the Levi–Civita connection alone. Facts:

- Weyl tensor of Riemannian curvature is of Petrov type D and coincides with that of Kerr.
- Tracefree Ricci tensor of Riemannian curvature is non-zero.
- Weyl tensor of Riemann–Cartan curvature vanishes identically.

It is quite interesting to study the properties of the underlying Riemannian geometry. We see that the interpretation of the curvature pieces is radically different in the Riemann–Cartan geometry.

Summary and open problems

For a wide class of Lagrangians quadratic in torsion and curvature there exist exact solutions that resemble the Kerr–(A)dS geometry. They come with localized null torsion proportional to the mass.

Questions that I would like to be able to answer:

- Can the null torsion be written in terms of the principal null congruence?
- Does the presence of torsion allow the existence of hidden symmetries?

Other avenues:

- Do the equations describing a particle in this spacetime separate? Under which conditions?

Thank you for your attention.

Abstract

The family of Kerr-NUT-(A)dS geometries in the context of Einstein's General Relativity possesses many interesting properties, most notably the existence of hidden symmetries encoded by a closed, non-degenerate conformal Killing–Yano 2-form. It is of considerable interest to study whether those properties extend to generalizations beyond General Relativity.

In this talk I will focus on a framework of theories that allows for non-vanishing torsion as well as for curvature, dubbed **Poincaré gauge gravity**. This class of theories, given a Lagrangian that is quadratic in torsion and curvature, possesses an exact solution that in many ways resembles the Kerr-(A)dS black hole of General Relativity.

I will present the main properties of this exact solution, which will provide the first steps towards understanding the role of hidden symmetries in the context of theories with non-vanishing torsion.