Differential forms:
from classical force to the Wilson loop

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PSI seminar
Perimeter Institute for Theoretical Physics

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Motivation & Outline

Differential forms provide a superior formalism as compared to vector calculus.

But: we can also use it to understand physics more easily.

I. Differential forms in 3D Euclidean space

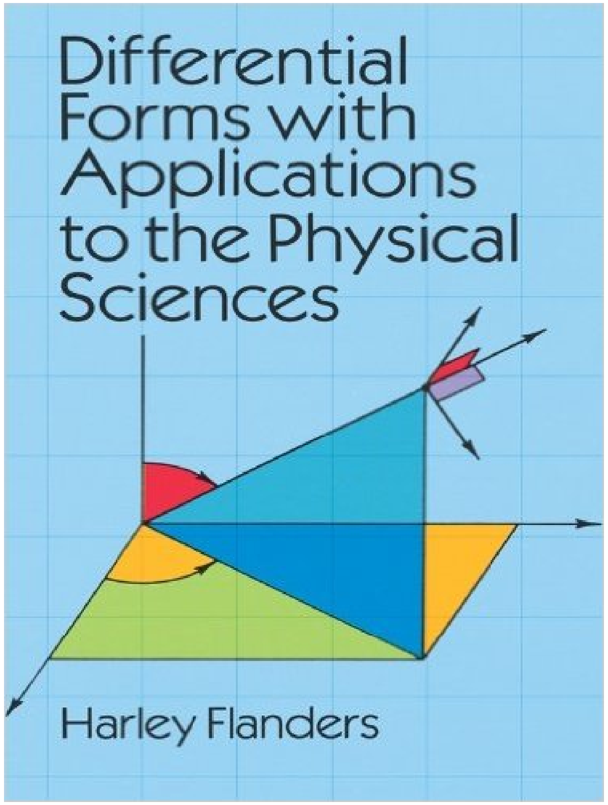
- Hodge dual
- visualization of forms
- example 1: classical force
- example 2: vacuum electrodynamics

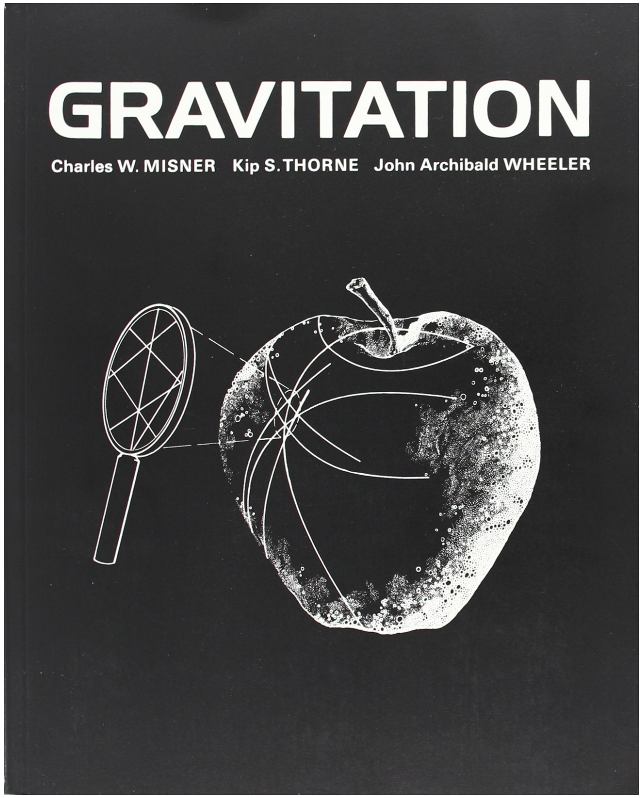
II. Differential forms in gauge theory

- electrodynamics as a U(1) gauge theory
- Stokes' theorem and the Wilson loop
- General Relativity and beyond

III. Conclusions & Summary

Literature


Differential Forms with Applications to the Physical Sciences
Harley Flanders


GRAVITATION
Charles W. MISNER Kip S. THORNE John Archibald WHEELER

I. Differential forms in 3D Euclidean space

A p-form in n dimensions has $\# = \frac{n!}{p!(n-p)!}$ independent components.

For n=3: **one** 0-form, **three** 1-forms, **three** 2-forms, **one** 3-form, and that's it.

1	dx	$dx \wedge dy$	$dx \wedge dy \wedge dz$
	dy	$dx \wedge dz$	
	dz	$dy \wedge dz$	

Note that there is a correspondence between p- and (n-p)-forms via the **Hodge** dual:

$$\star\,\omega = \star\left(\frac{1}{p!}\omega_{a_1\ldots a_p} dx^{a_1} \wedge \cdots \wedge dx^{a_p}\right) = \frac{1}{p!(n-p)!}\omega^{a_1\ldots a_p}\epsilon_{a_1\ldots a_p b_1 \ldots b_{n-p}} dx^{b_1} \wedge \cdots \wedge dx^{b_{n-p}}$$

In 3D one has $\star\,\star = 1$, and therefore

$$dx \simeq dy \wedge dz, \quad dy \simeq dz \wedge dx, \quad dz \simeq dx \wedge dy, \quad 1 \simeq dx \wedge dy \wedge dz.$$

Visualization of differential 1-forms (here: in 3D)

How does a 1-form act on a vector? Simple counting: $\partial_\mu \lrcorner \, \mathrm{dx}^\alpha = \delta^\alpha_\mu$

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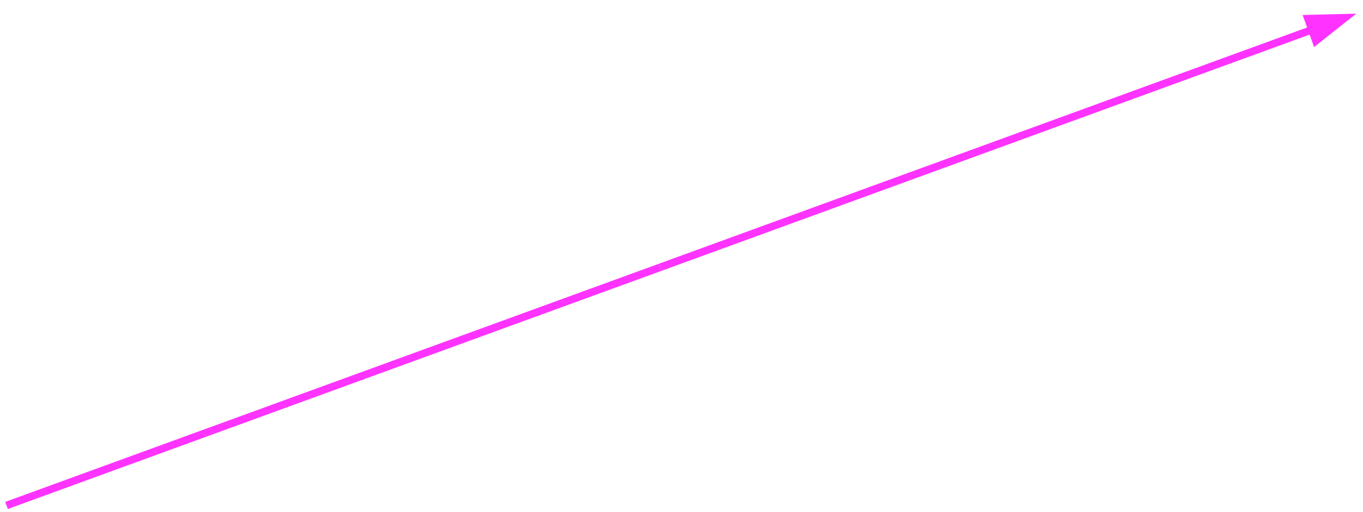

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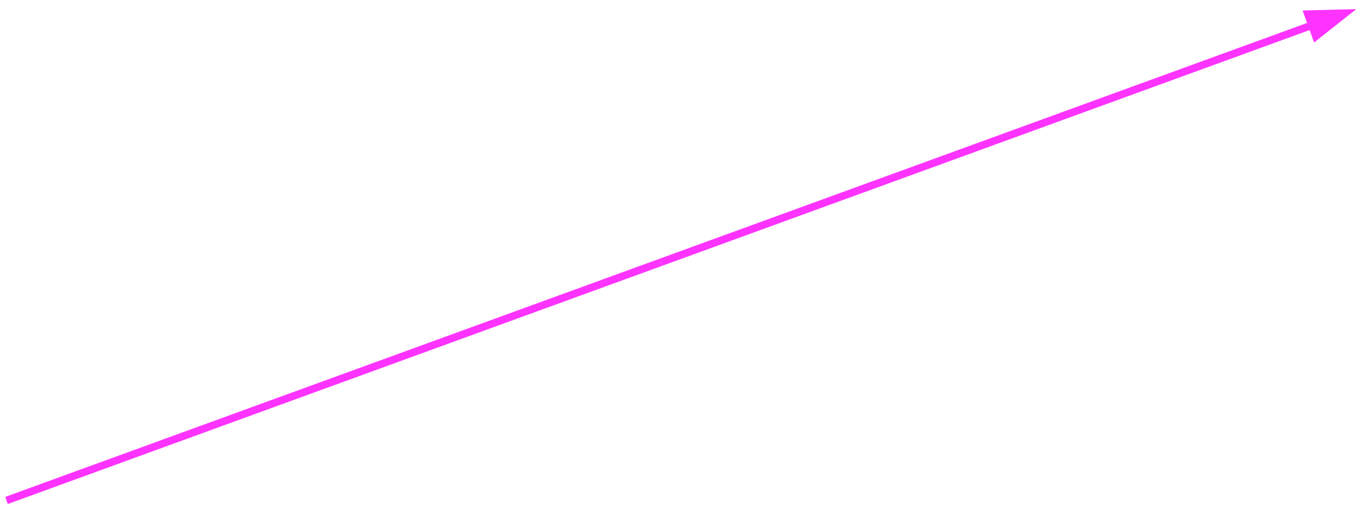

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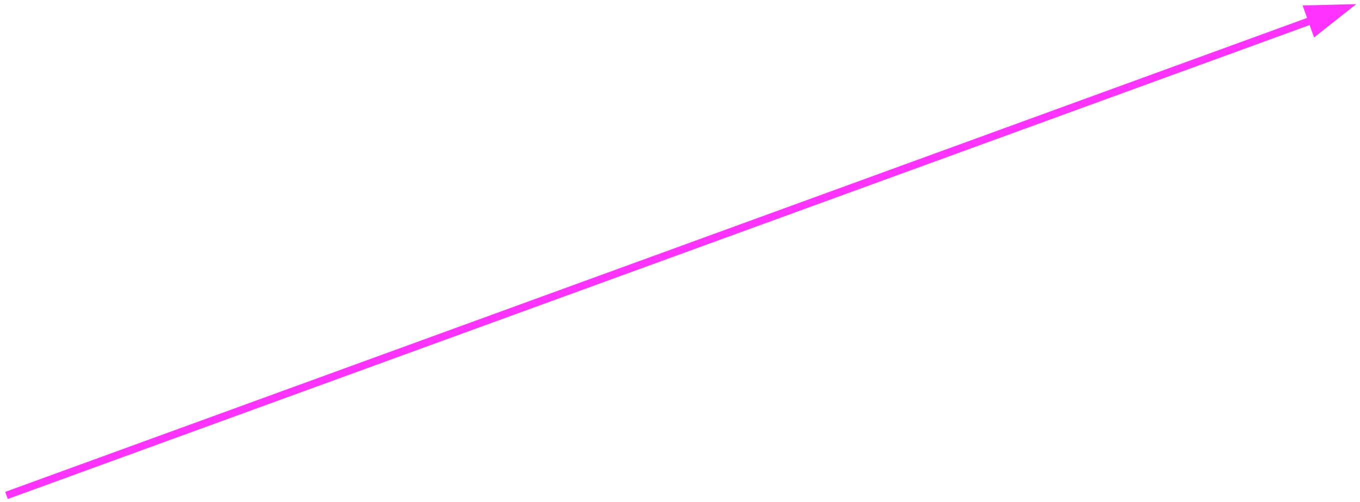

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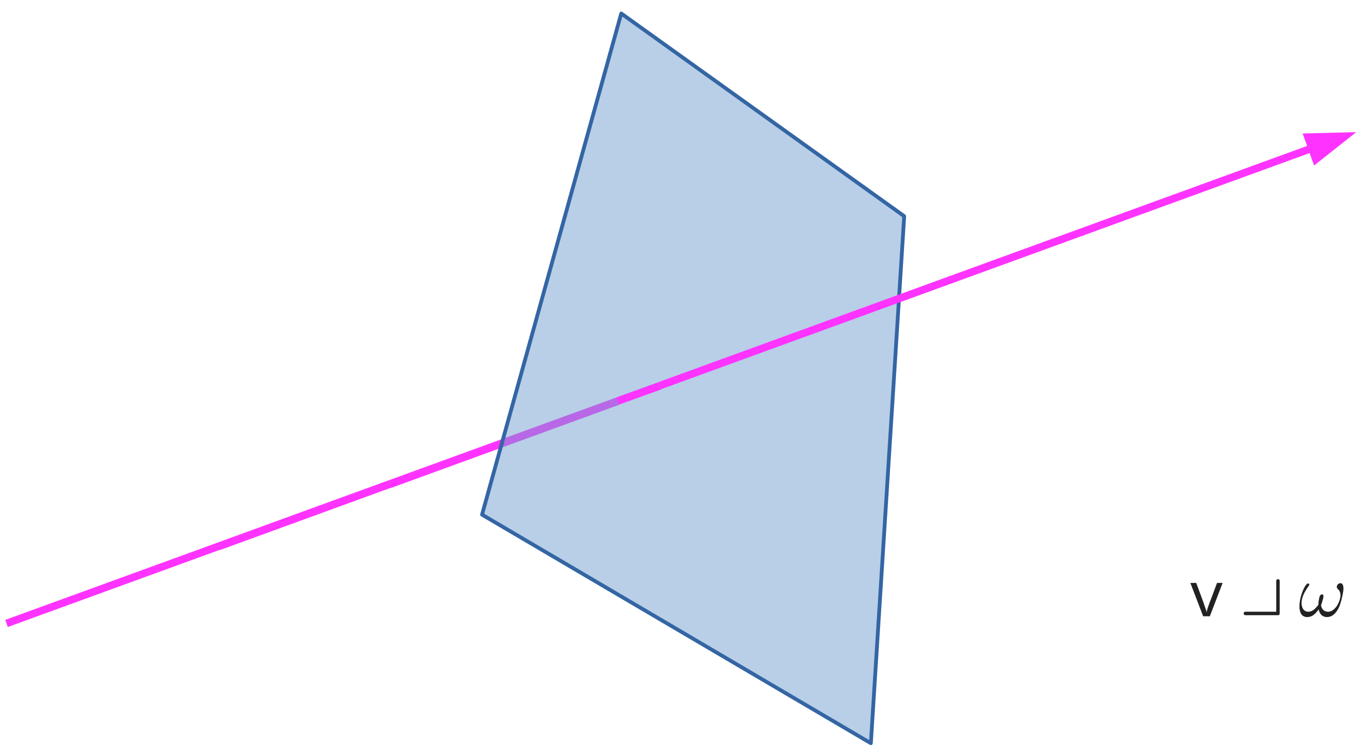

$\mathrm{v} \lrcorner \, \omega = 1$

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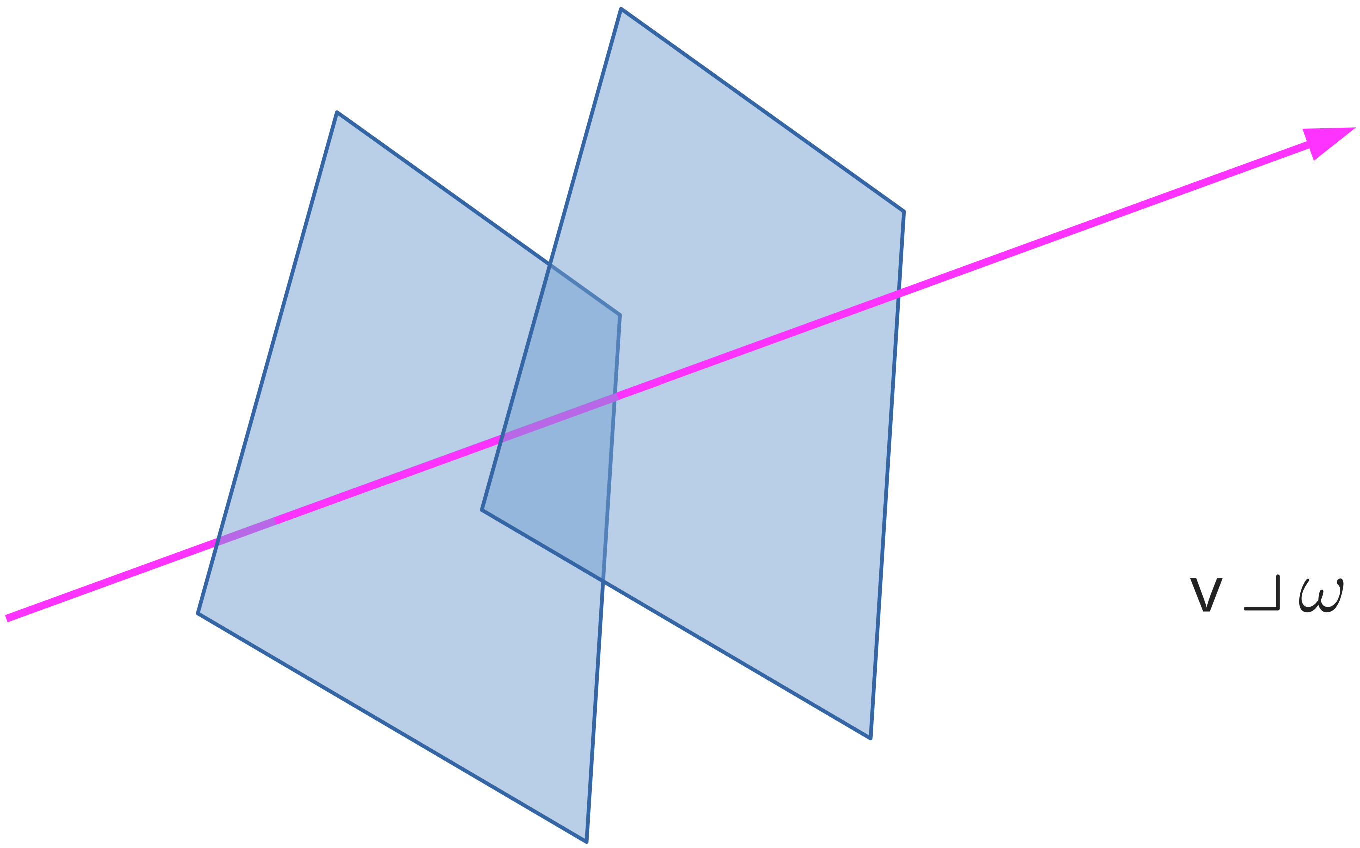

$v \lrcorner \, \omega = 2$

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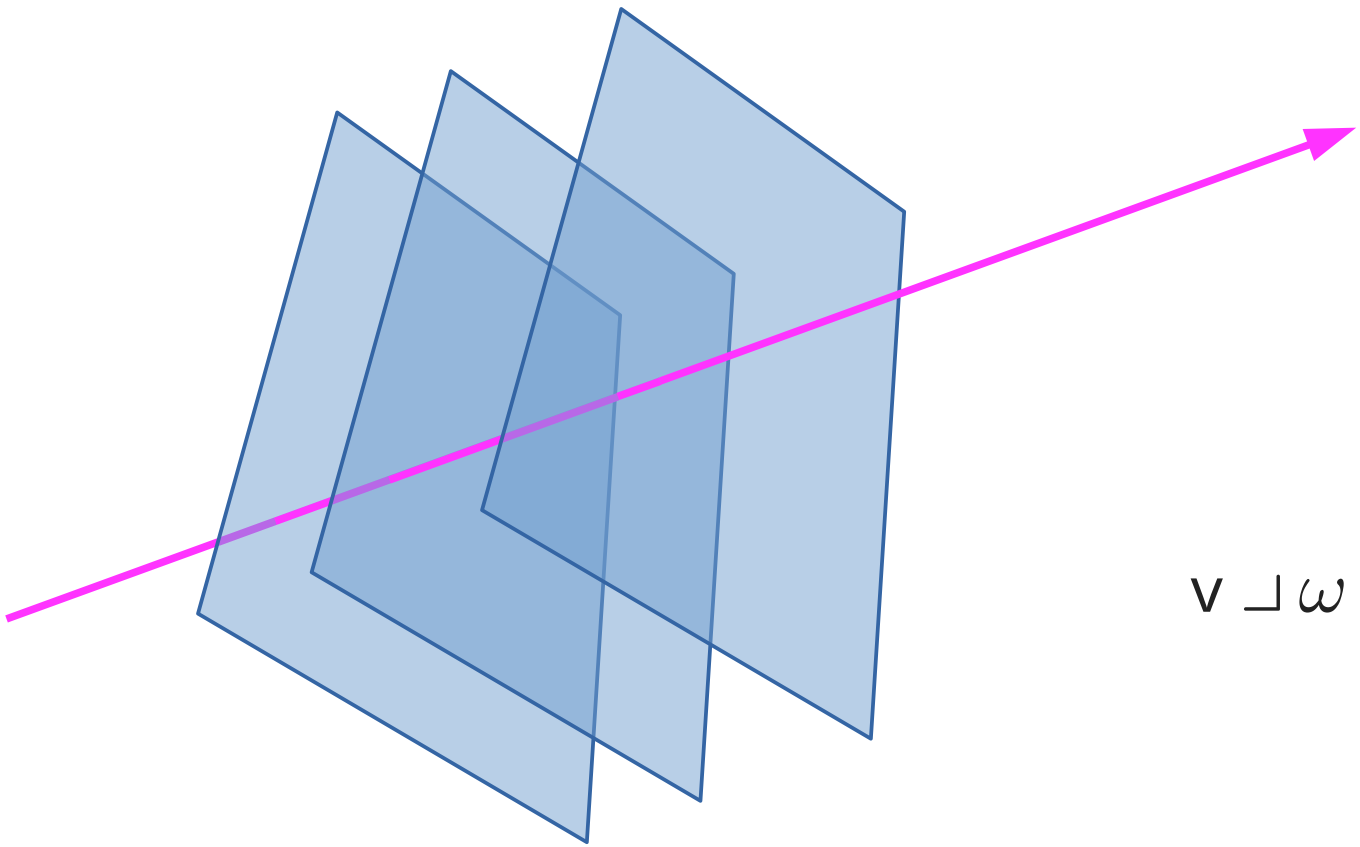

$v \lrcorner \, \omega = 3$

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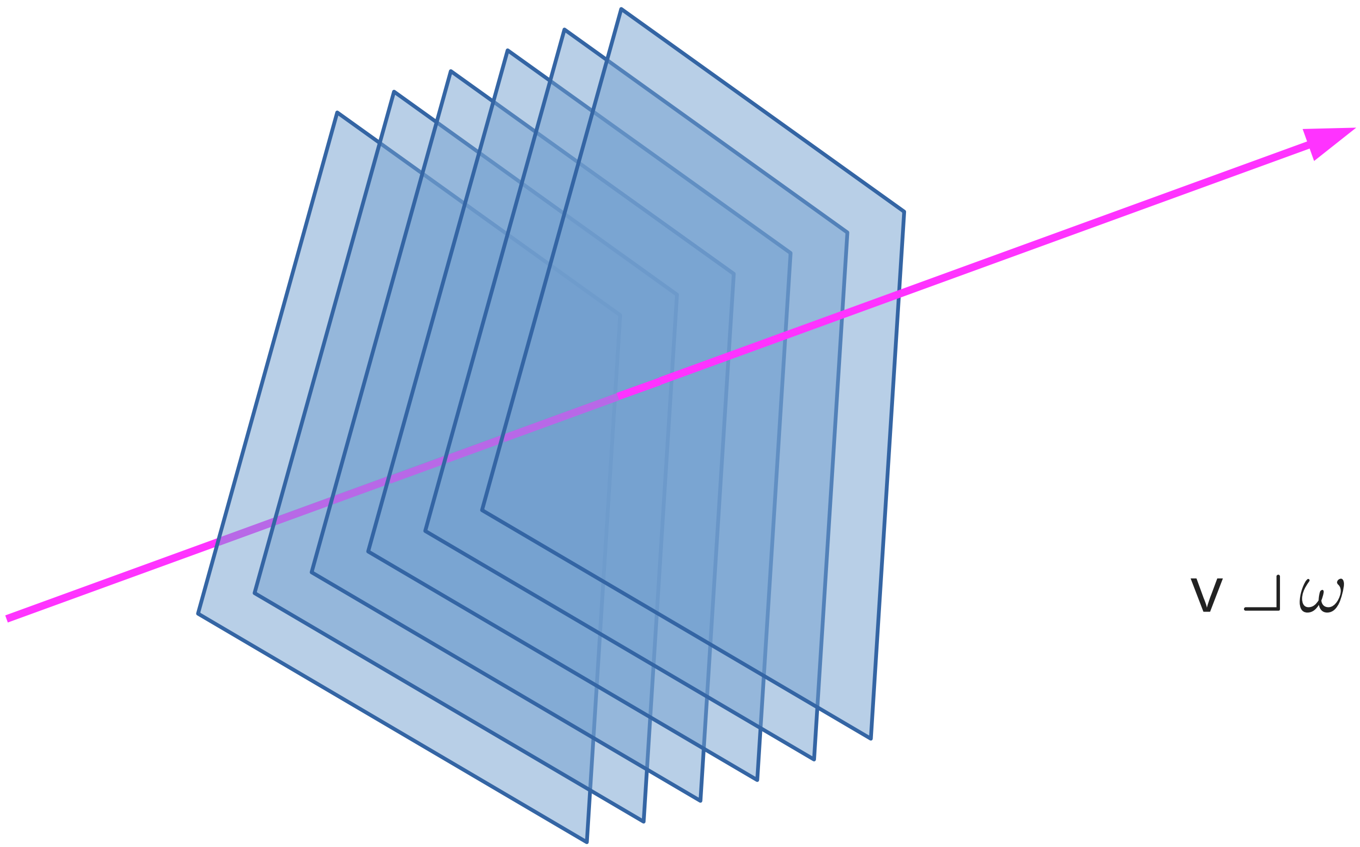
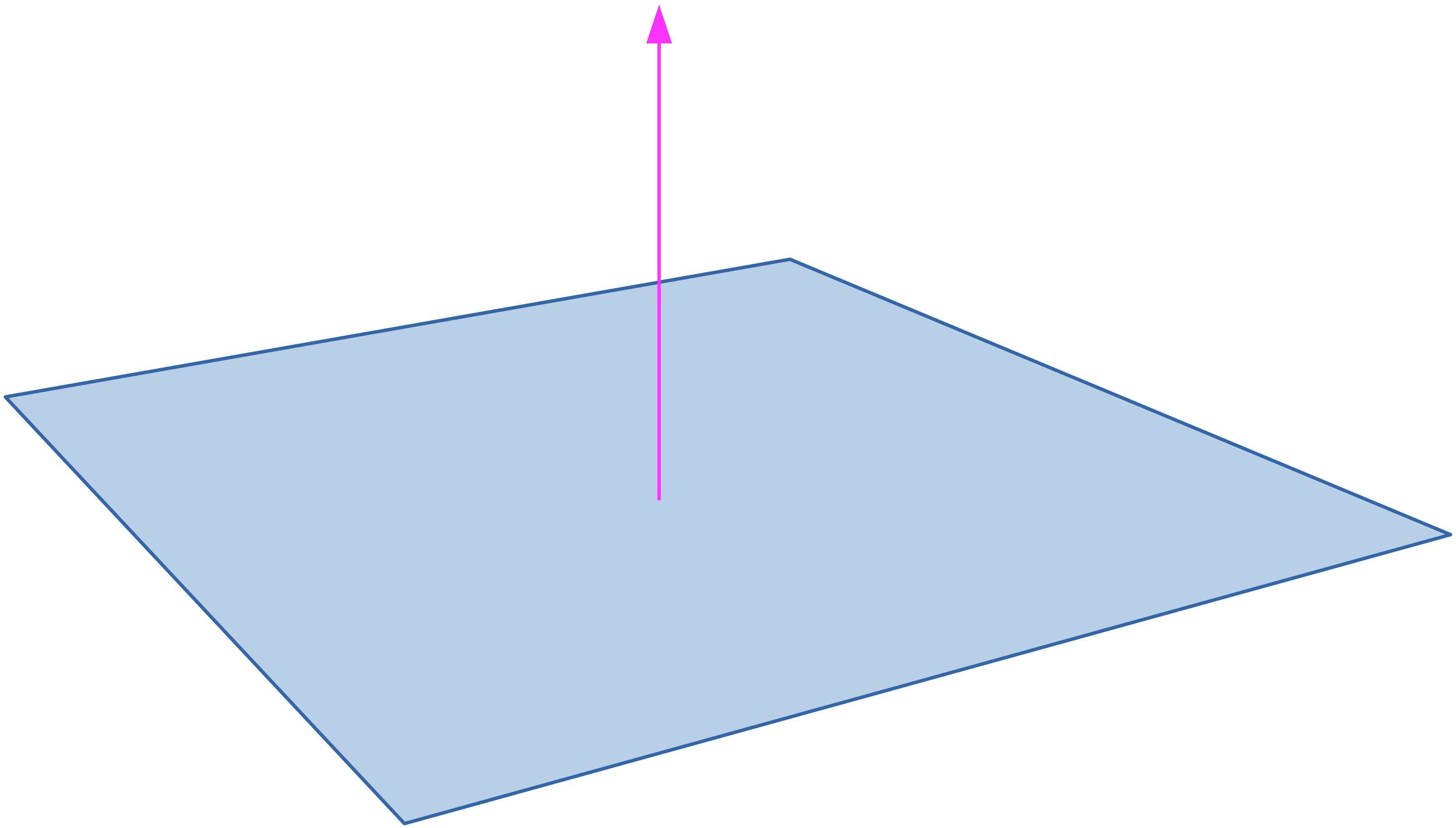


$v \lrcorner \, \omega = 6$

Visualization of differential 1-forms (here: in 3D)

Why does that work?

→ a (n-1)-dimensional hypersurface is described by an n-dimensional normal vector.



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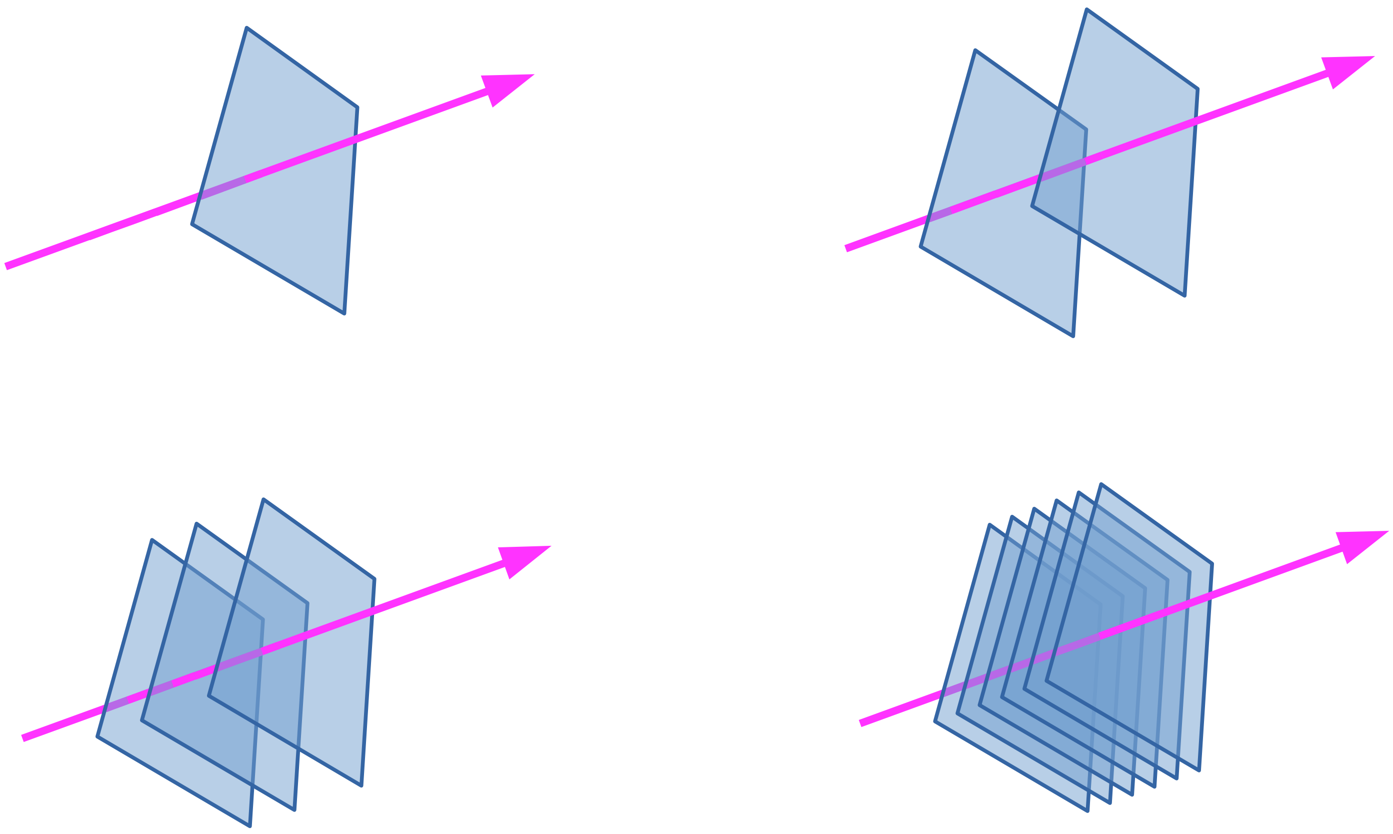
So which form is the “strongest” one?



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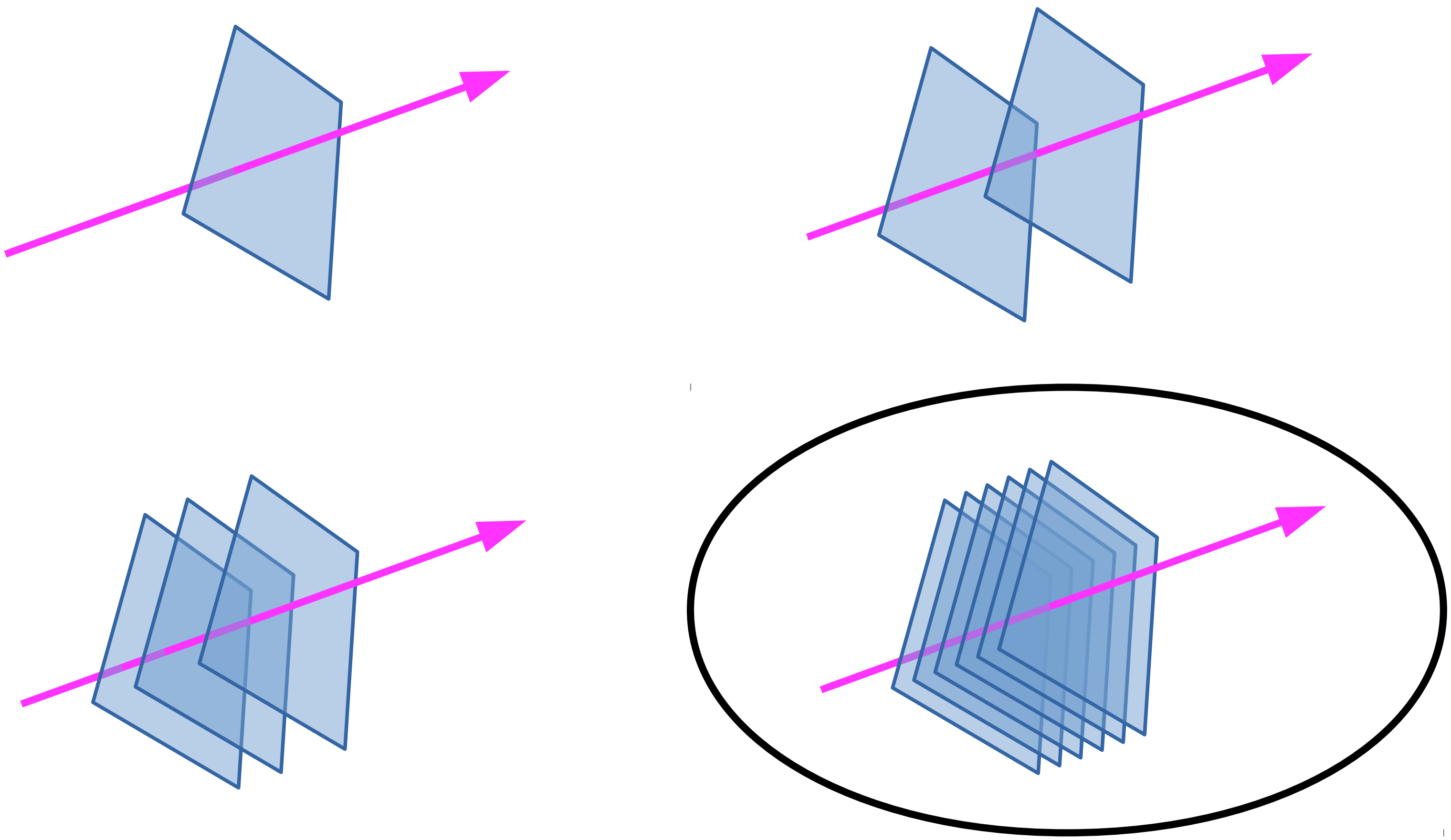

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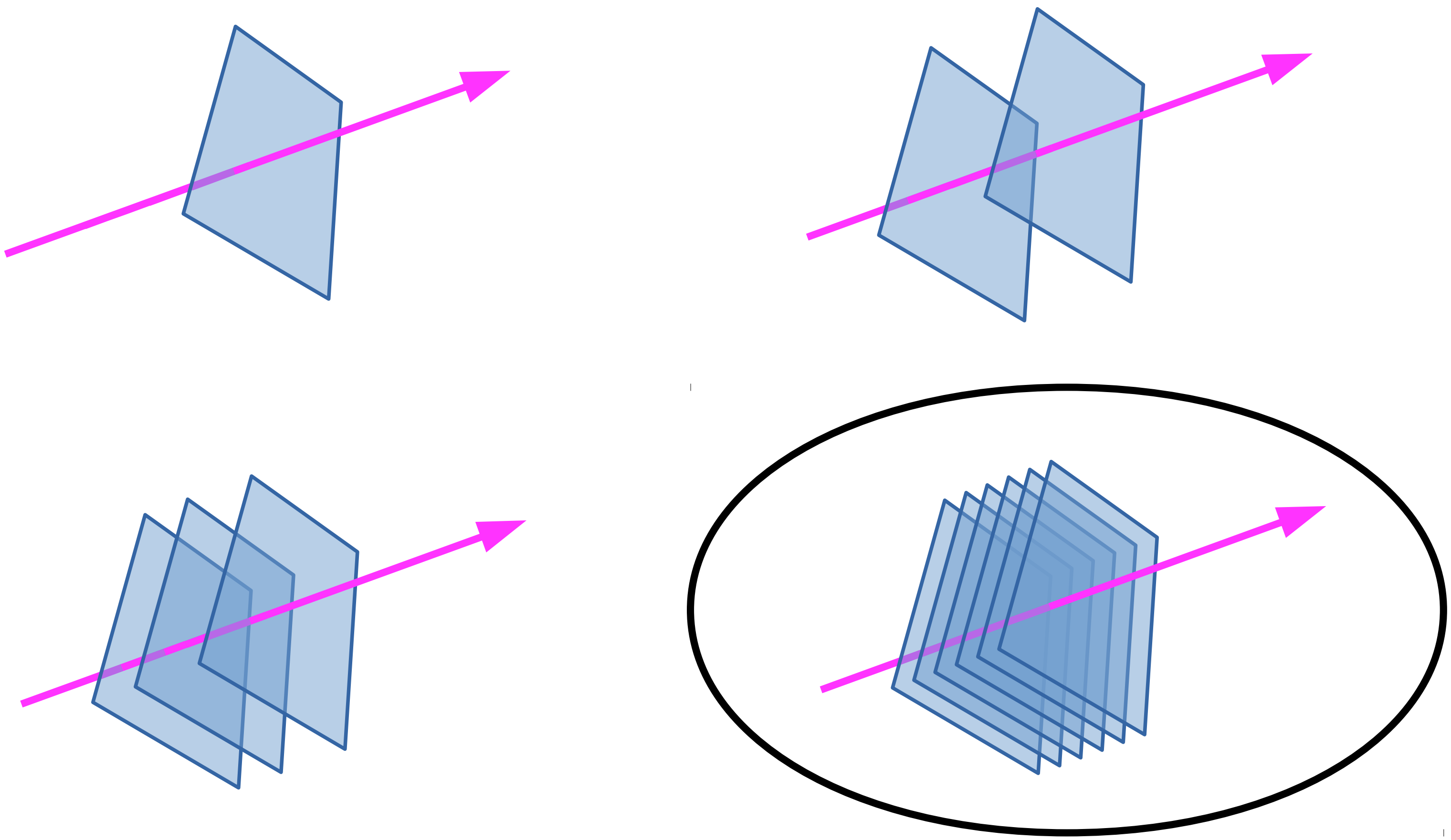

Physical interpretation: work along the direction specified by vector.

Visualization of differential 1-forms: a word of caution

Explicit construction can be very hard, for example:

$$\omega = 12y\mathrm{d}x - \sin(y)\mathrm{d}y + zx^2\mathrm{d}z = f(x,y,z)\mathrm{d}x + g(x,y,z)\mathrm{d}y + h(x,y,z)\mathrm{d}z$$

1. Corresponding vector field is $v = f\partial_x + g\partial_y + h\partial_z$, OK.

2. Can we derive a potential? Only possible for $\mathrm{d}\omega = 0 \Leftrightarrow \nabla \times v = 0$.

   - YES: Very good, see before. Differential form = equipotential surfaces.
   - NO: More complicated, but surfaces can still be built locally.

→ Bottom line: locally, the surface picture is reasonable,
but be careful with global concepts.

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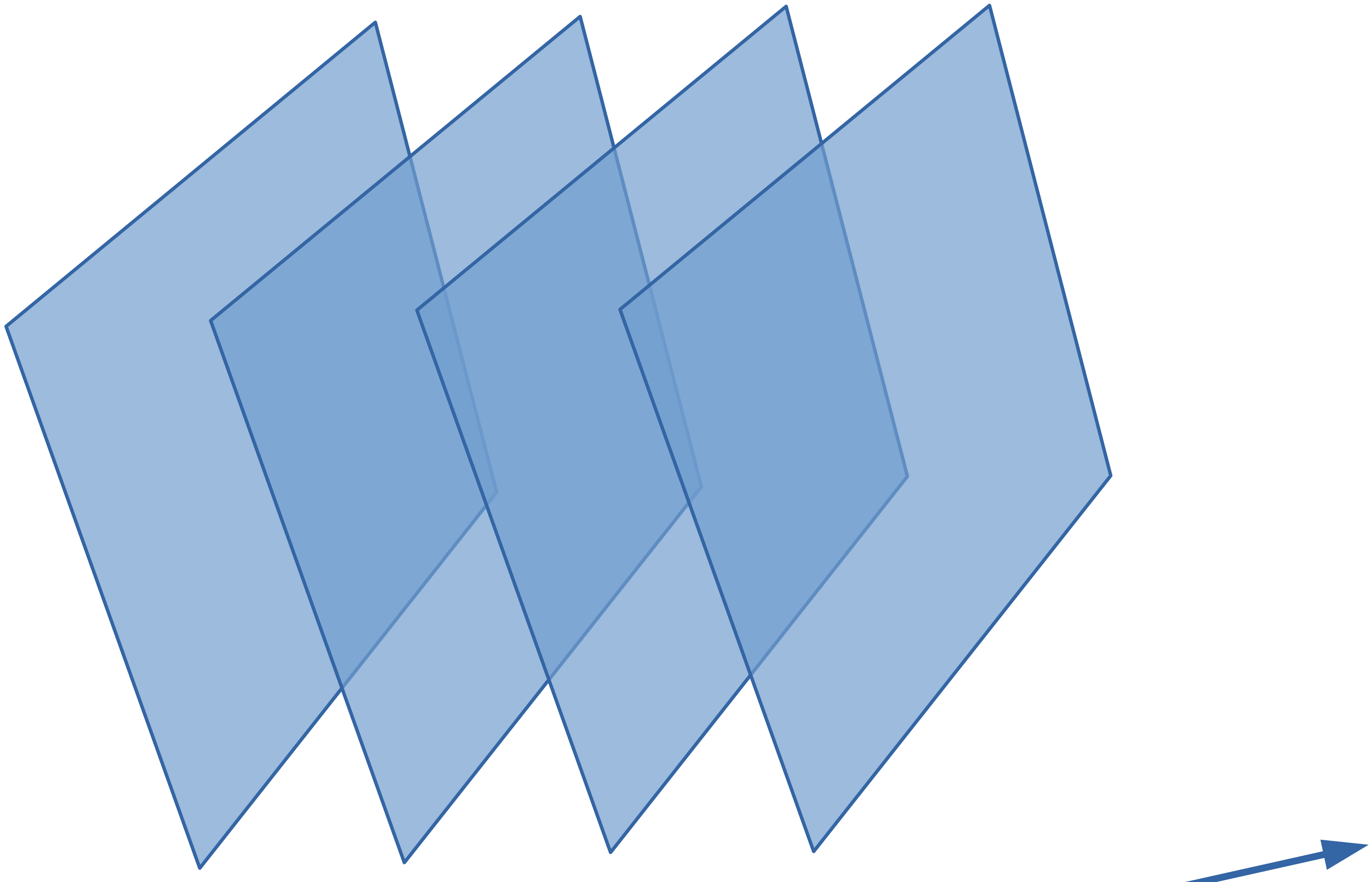

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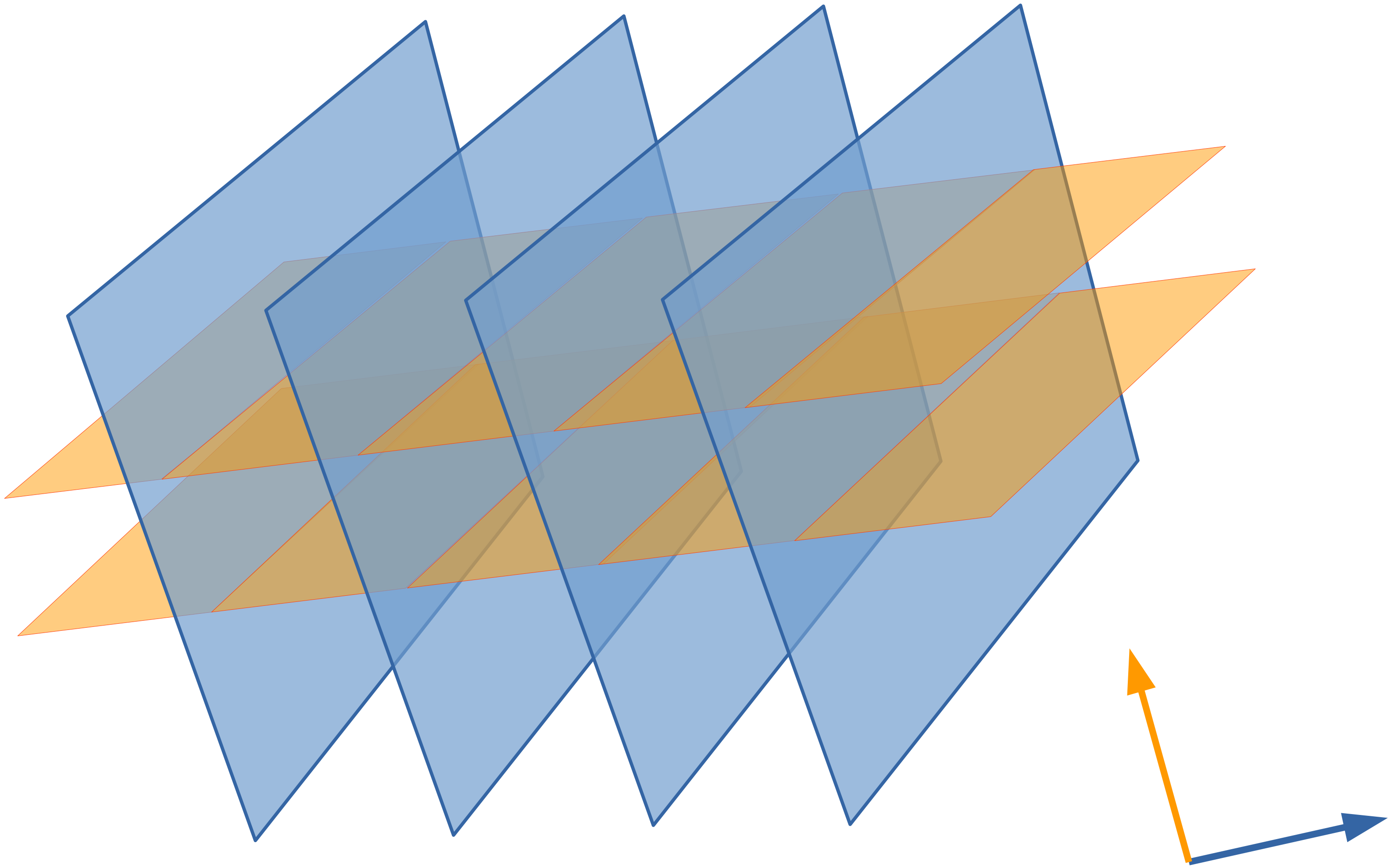

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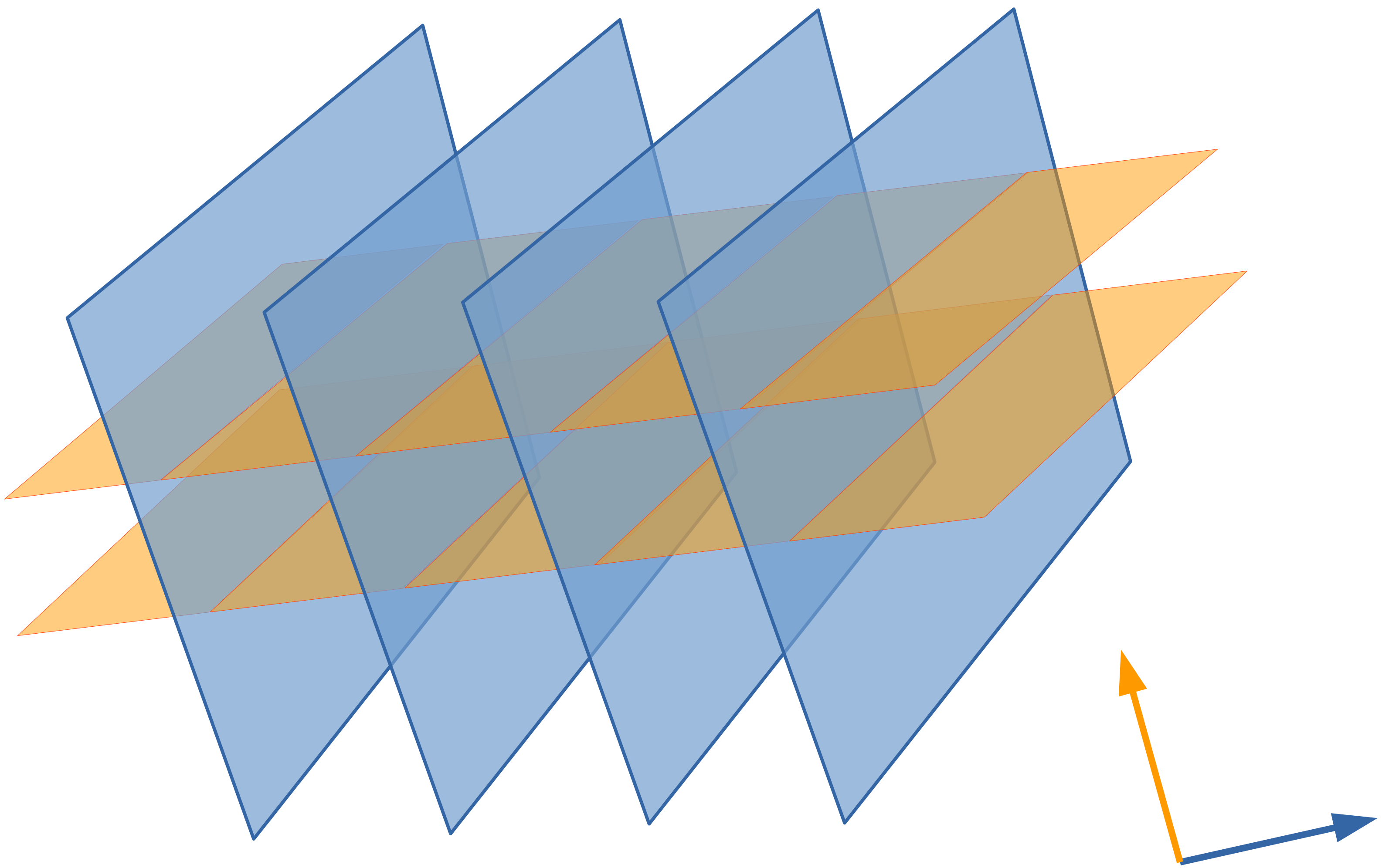

Physical interpretation: **flux** through area spanned by the two vectors.

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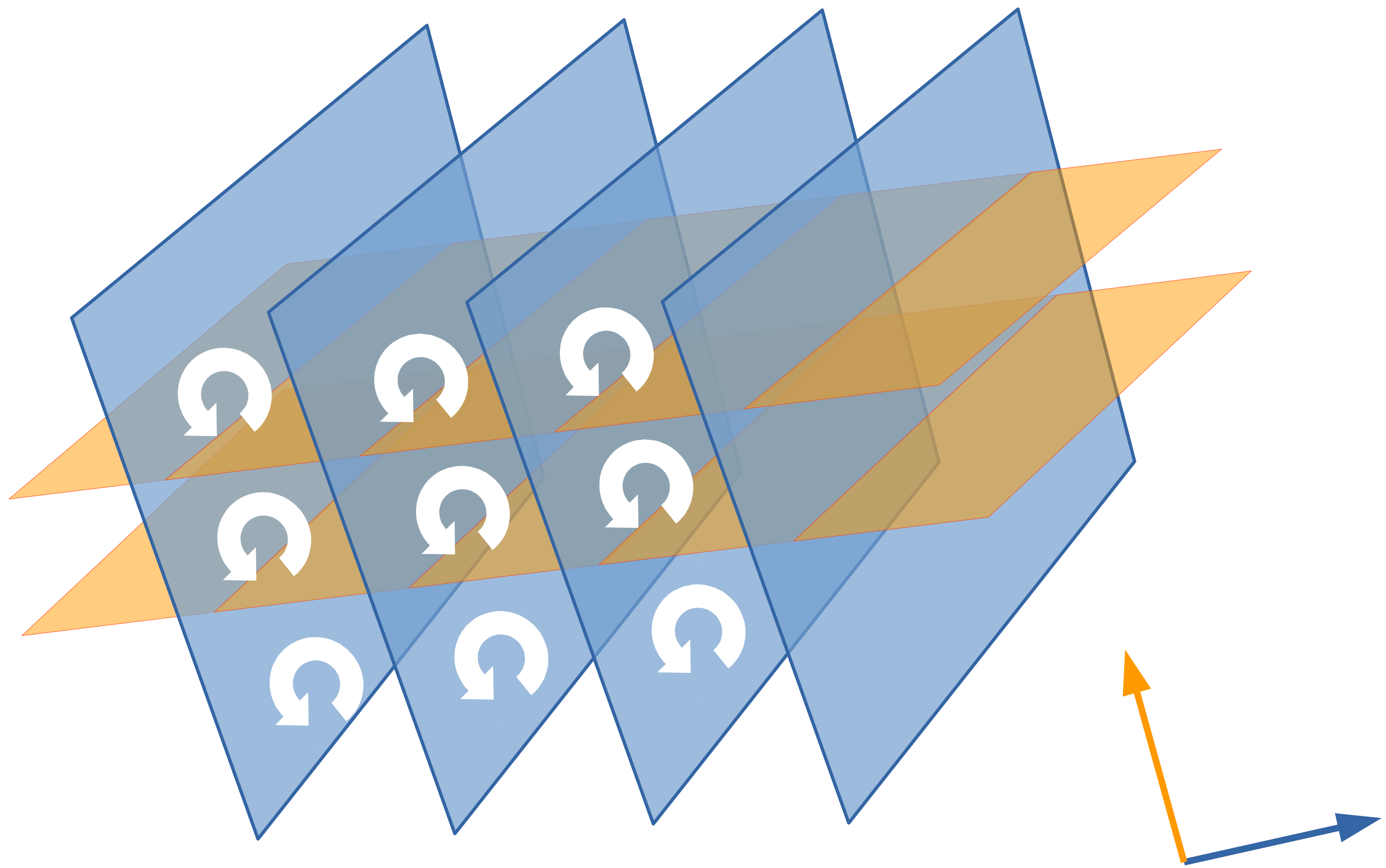

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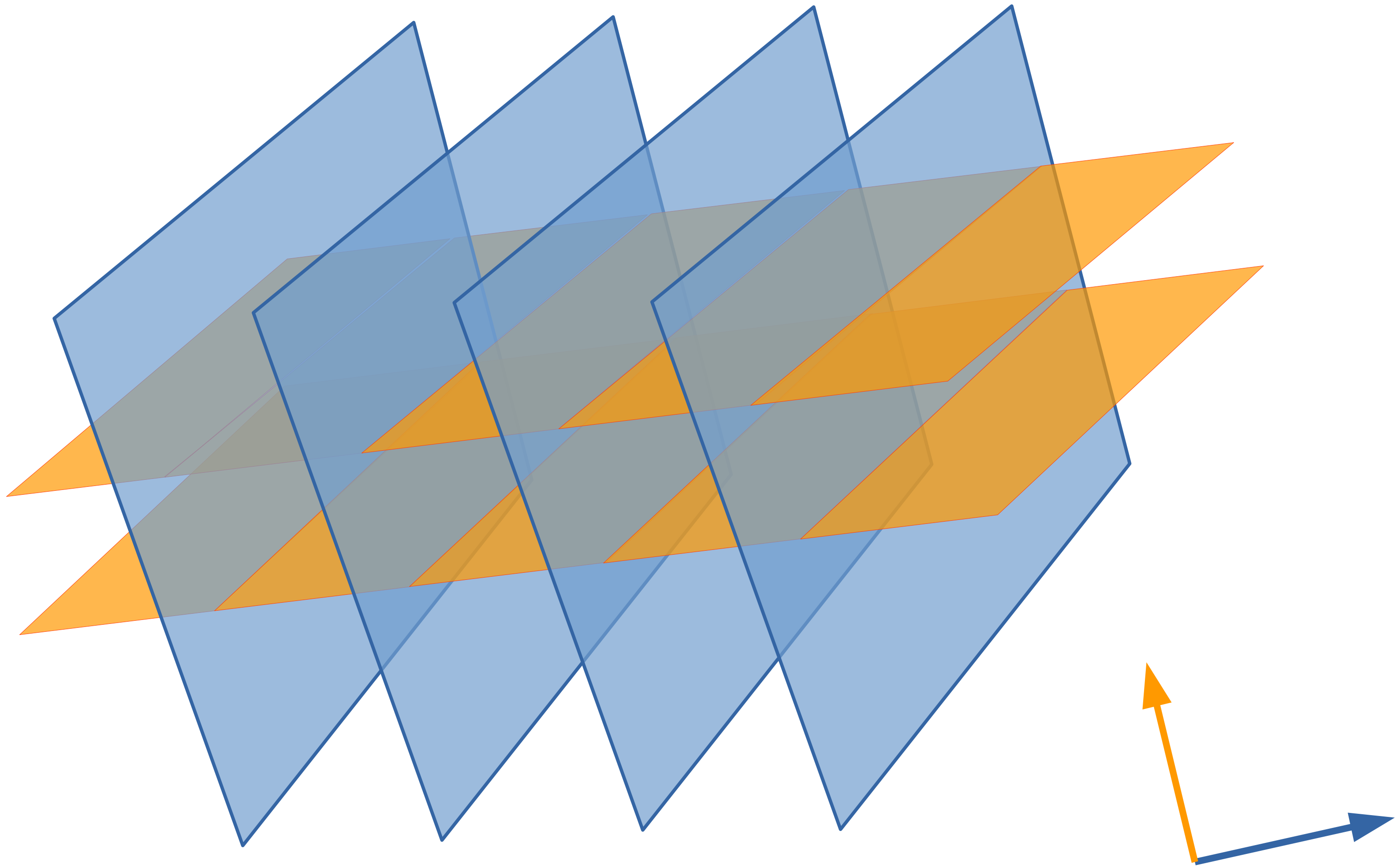

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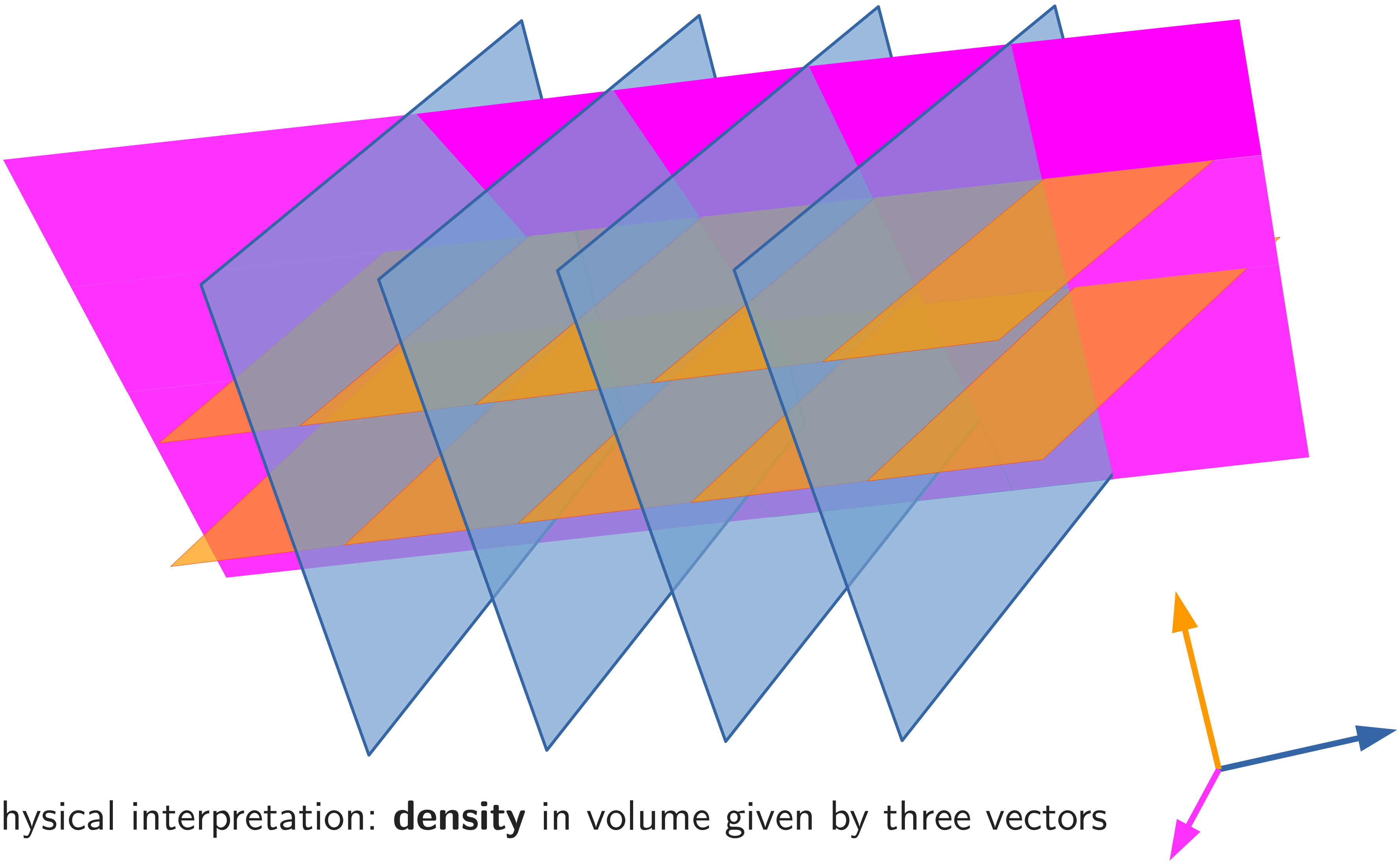
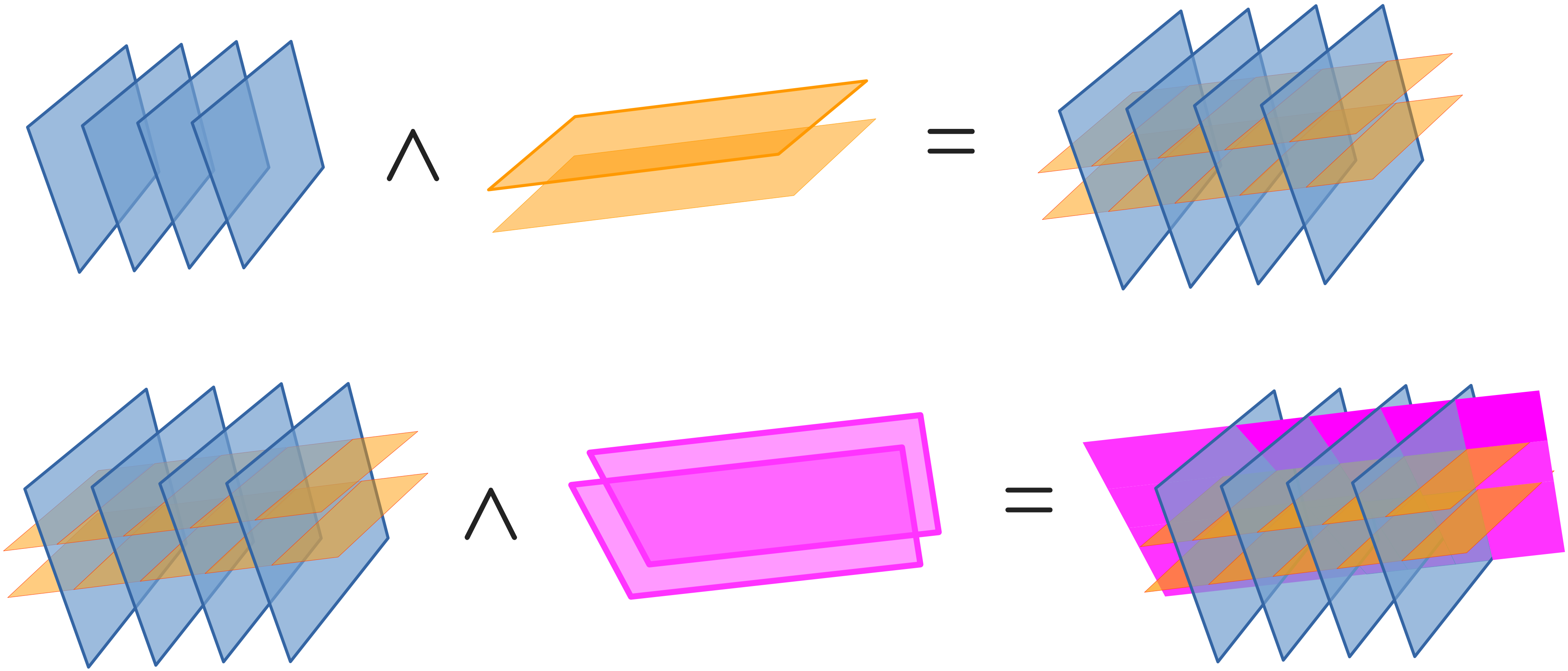


Physical interpretation: **density** in volume given by three vectors

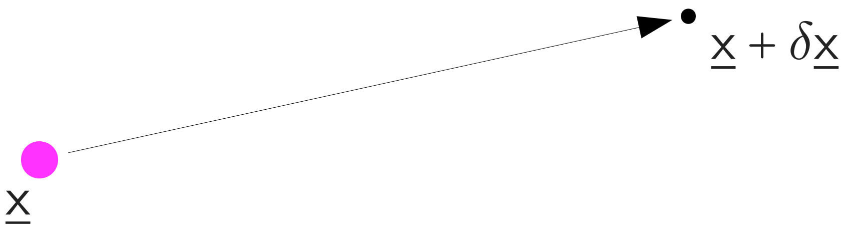
In summary:

The exterior product visualized:



Example 1: classical force

Imagine you want to determine a **force** $\underline{F}$ on a particle :



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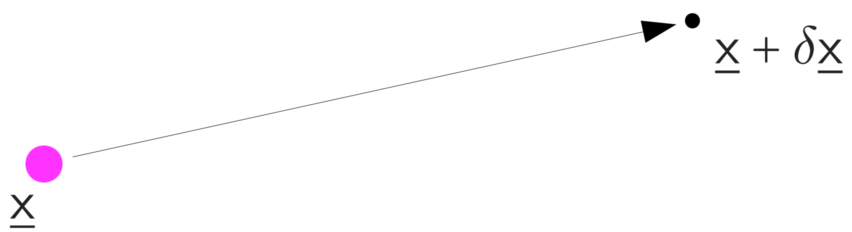

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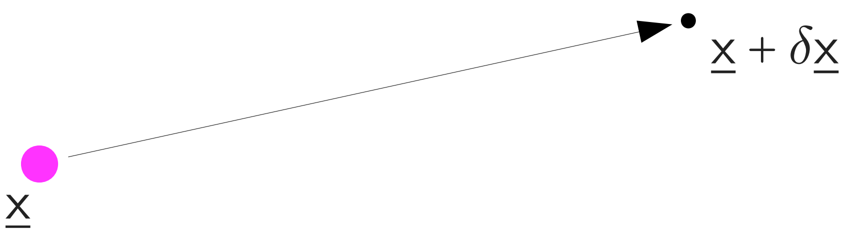

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$\rightarrow$ **operational interpretation** of differential forms?

Example 2: vacuum electrodynamics

The field strength 2-form is given by

$$F = dt \wedge \underbrace{(E_x dx + E_y dy + E_z dz)}_{\substack{\text{electric field} \\ \leftrightarrow \text{ 1-form}}} + \underbrace{B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy}_{\substack{\text{magnetic field} \\ \leftrightarrow \text{ 2-form}}} \,.$$

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This is equivalent to the antisymmetric $\binom{0}{2}$ tensor

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} .$$

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Translation formulas: $F = \frac{1}{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta$, $\quad F_{\mu\nu} = \partial_\nu \lrcorner (\partial_\mu \lrcorner F)$

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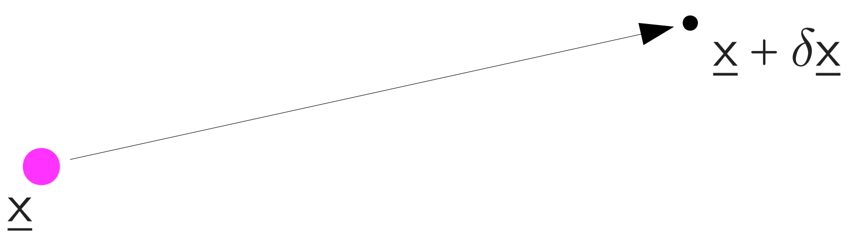
Operational interpretation of the **electric** field:



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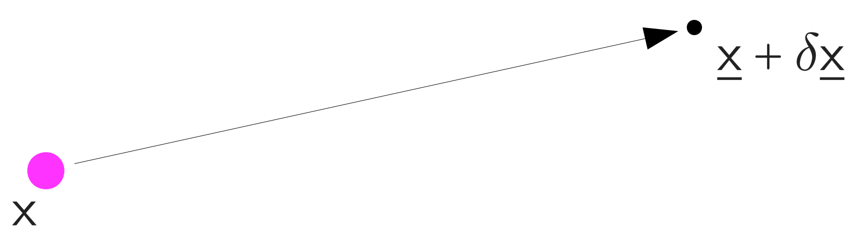

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In differential form language: $W = \delta x \lrcorner E$ $\;\longrightarrow$ equivalent to the classical force.

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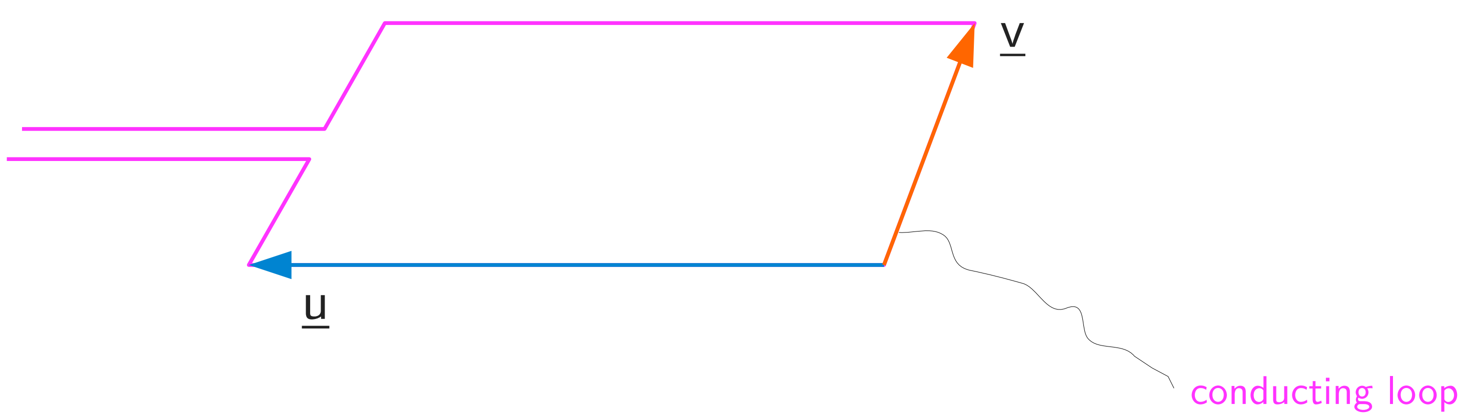
Operational interpretation of the **magnetic** field 2-form B:



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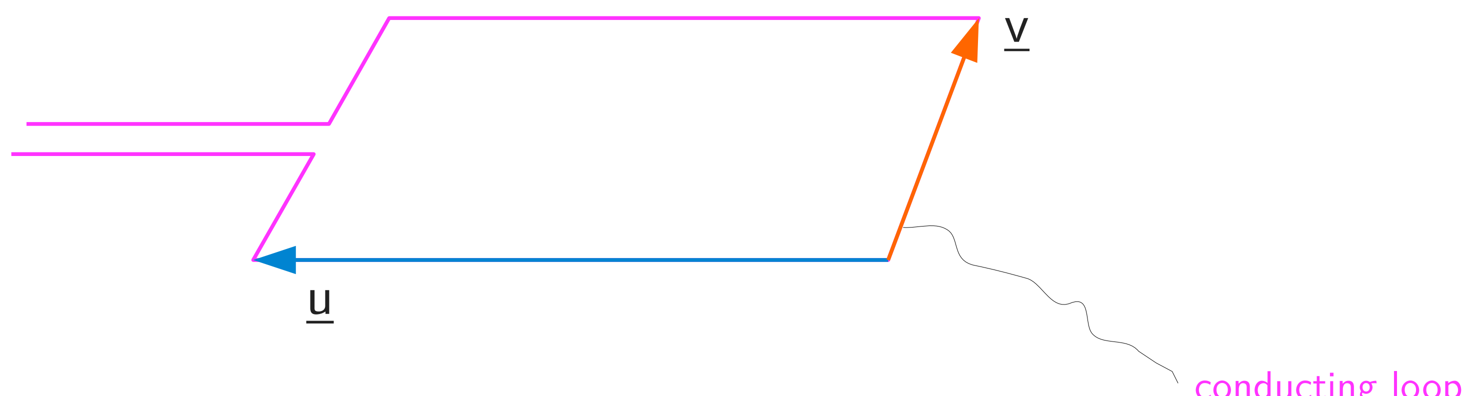
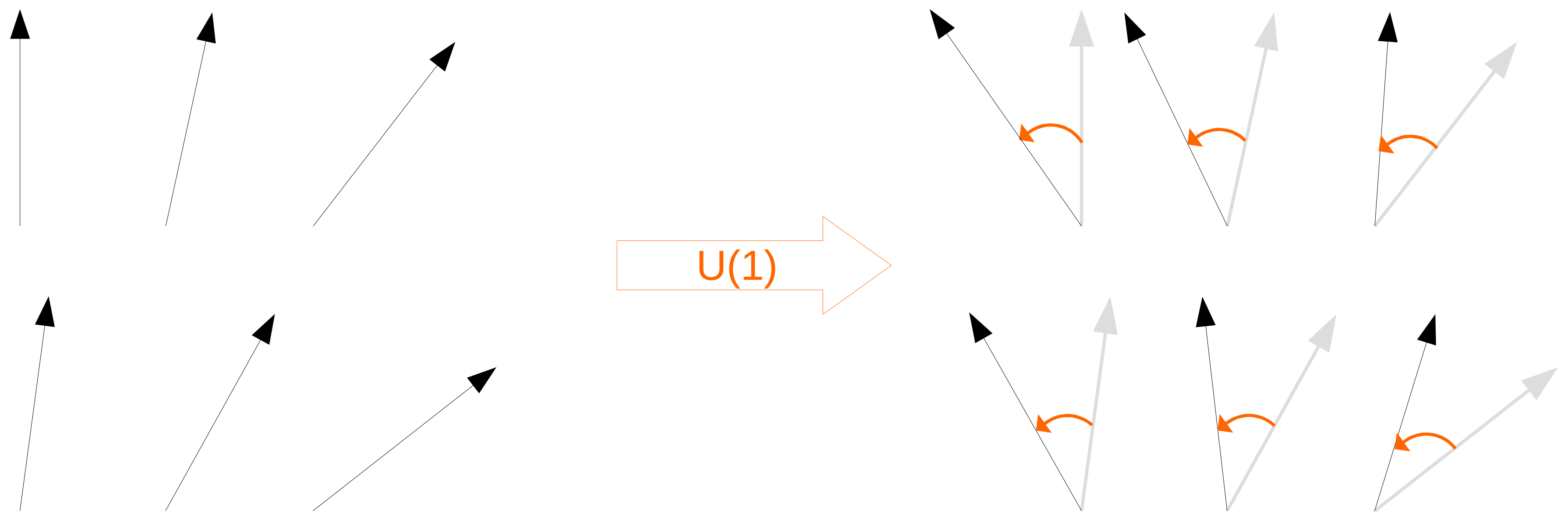
$$V_{ind} = -\partial_t \Phi$$



$\rightarrow$ magnetic flux $\Phi = u \lrcorner (v \lrcorner B)$

II. A brief introduction to U(1) gauge theory

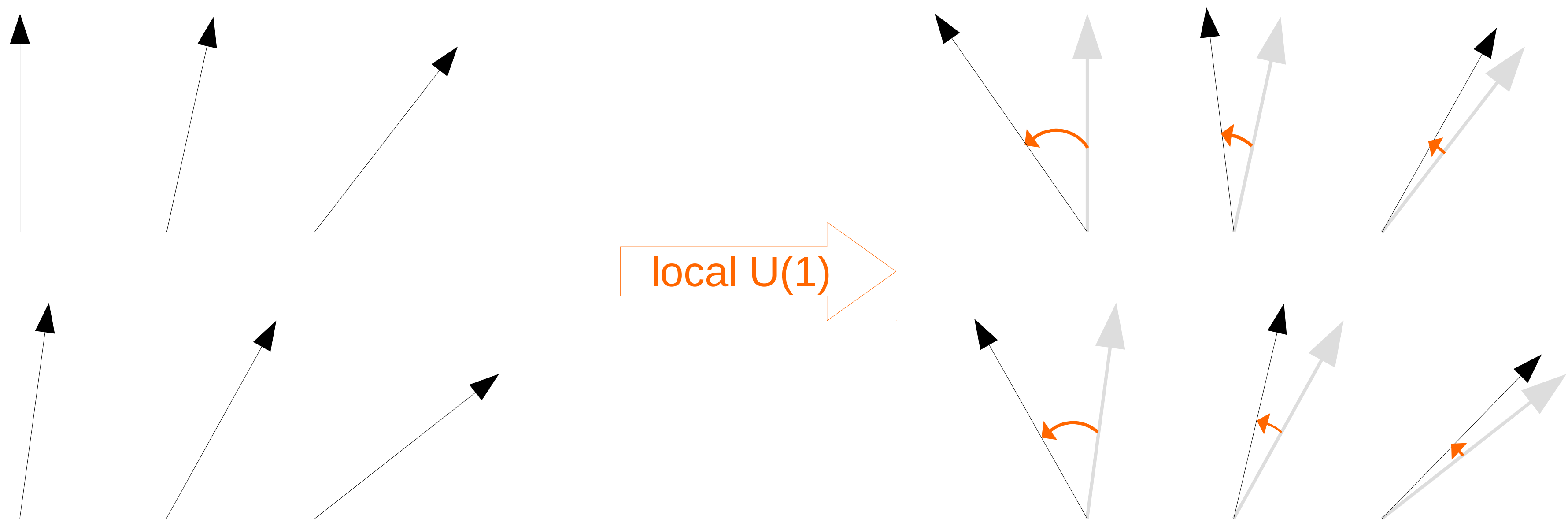
Consider a complex field ϕ under a global U(1) transformation $\phi \mapsto e^{i\alpha}\phi$, with $\alpha \in \mathbb{R}$:



If the theory is invariant under this transformation, we call U(1) a **rigid symmetry**.

II. A brief introduction to U(1) gauge theory

Now carry out a local transformation $\phi \mapsto e^{i\alpha(x)}\phi$:

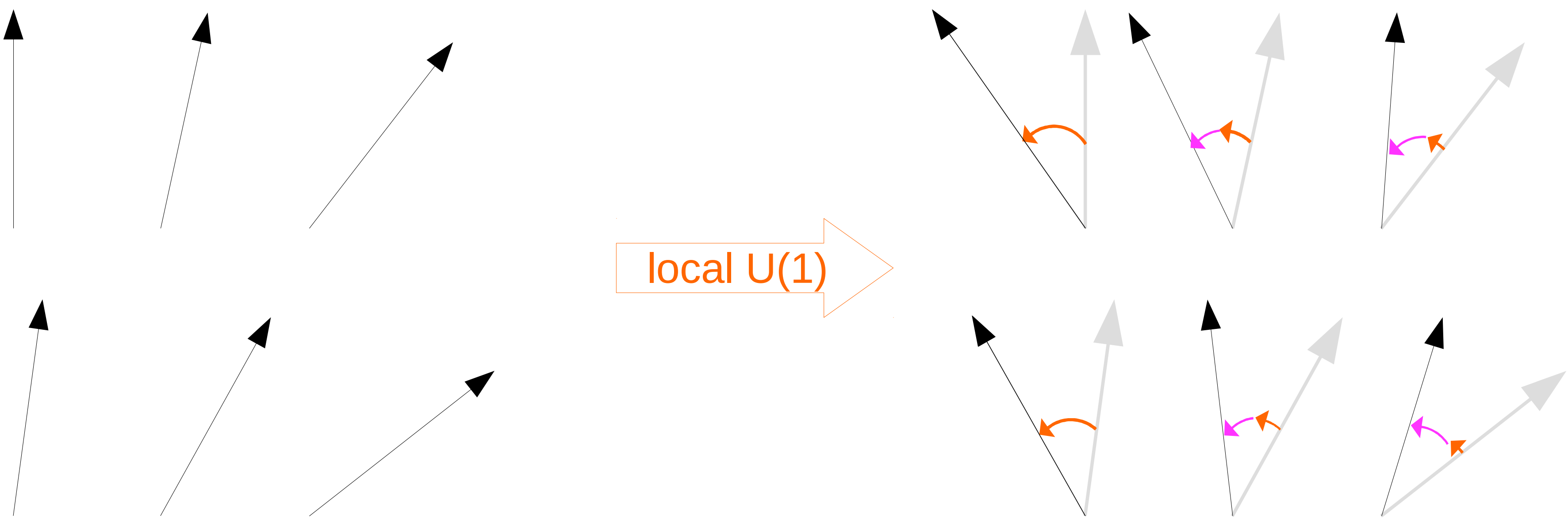


Due to x-dependence, any dynamical theory is **not invariant anymore**.

How do we rescue this? We need a gauge potential A!

II. A brief introduction to U(1) gauge theory

The gauge potential restores gauge invariance by $d \mapsto d + ie\,A$:

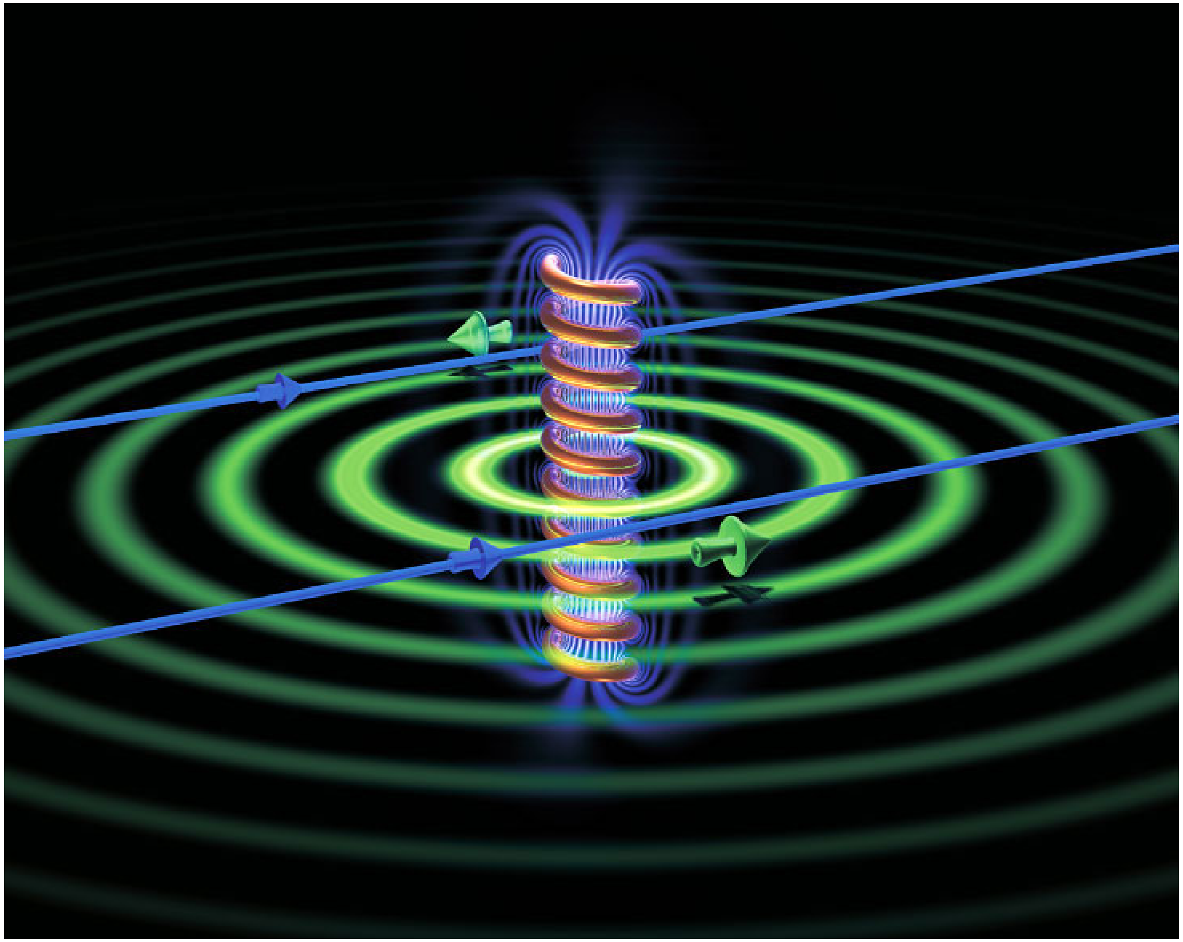


What have we gained? We can construct a Lagrangian for A using $F := dA$:

$\mathcal{L} := F \wedge \star F + j \wedge A$ yields **electrodynamics** with conserved current j

Electrodynamics: the Aharonov–Bohm effect

Consider the wave function $\Psi(x, t)$ of a particle which travels around a closed loop $\mathscr{C}$.



http://physics.aps.org/story/v28/st4

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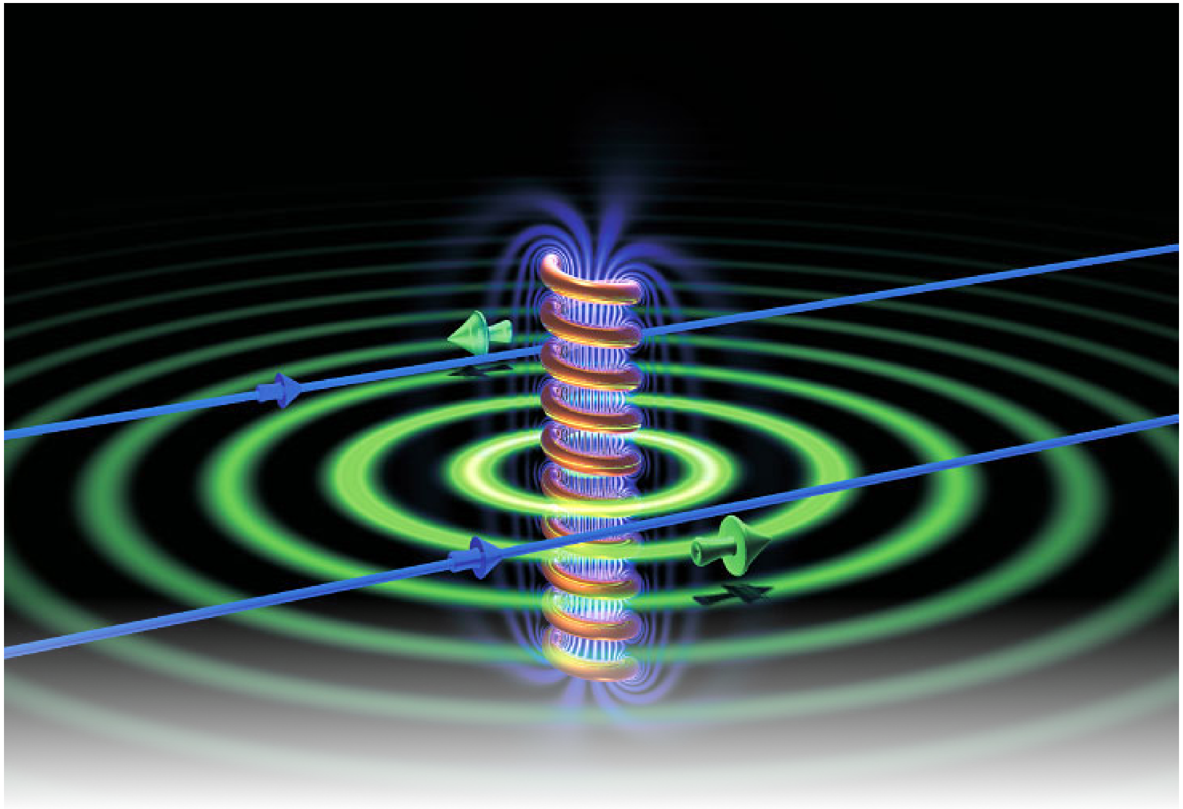

It picks up the phase shift $e^{i\phi}$, with $\phi \propto \oint_{\mathscr{C}} A$.

The path $\mathscr{C}$ is closed, and for a trivial topology this implies that $\mathscr{C} = \partial S$.

Stokes' theorem then gives us $\phi \propto \oint_{\partial S} A \propto \oint_{S} dA$ such that $e^{i\phi} \in U(1)$.

The Wilson loop

We saw: integrating the gauge connection around a closed loop gives a group element.

$$\rightarrow \text{Wilson loop } W_{\mathscr{C}} := \mathrm{Tr}\Big[\mathcal{P}\exp \mathrm{i}\oint_{\mathscr{C}} A\Big]$$

In gauge theories, $A = A^a t_a$, where t_a are the generators of the Lie group.

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Why is it interesting?

- Yang–Mills theories
- Loop Quantum Gravity
- but even in General Relativity, if you look for it...

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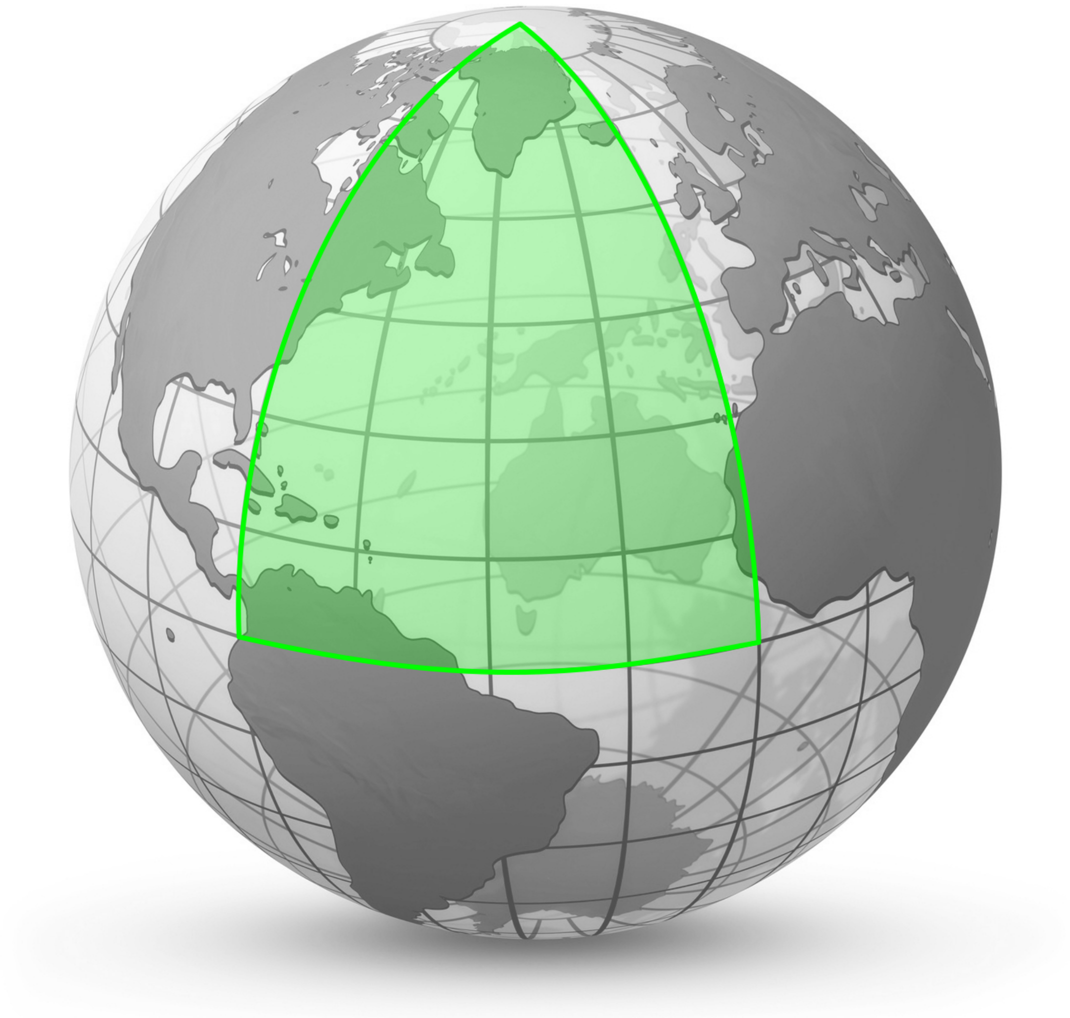

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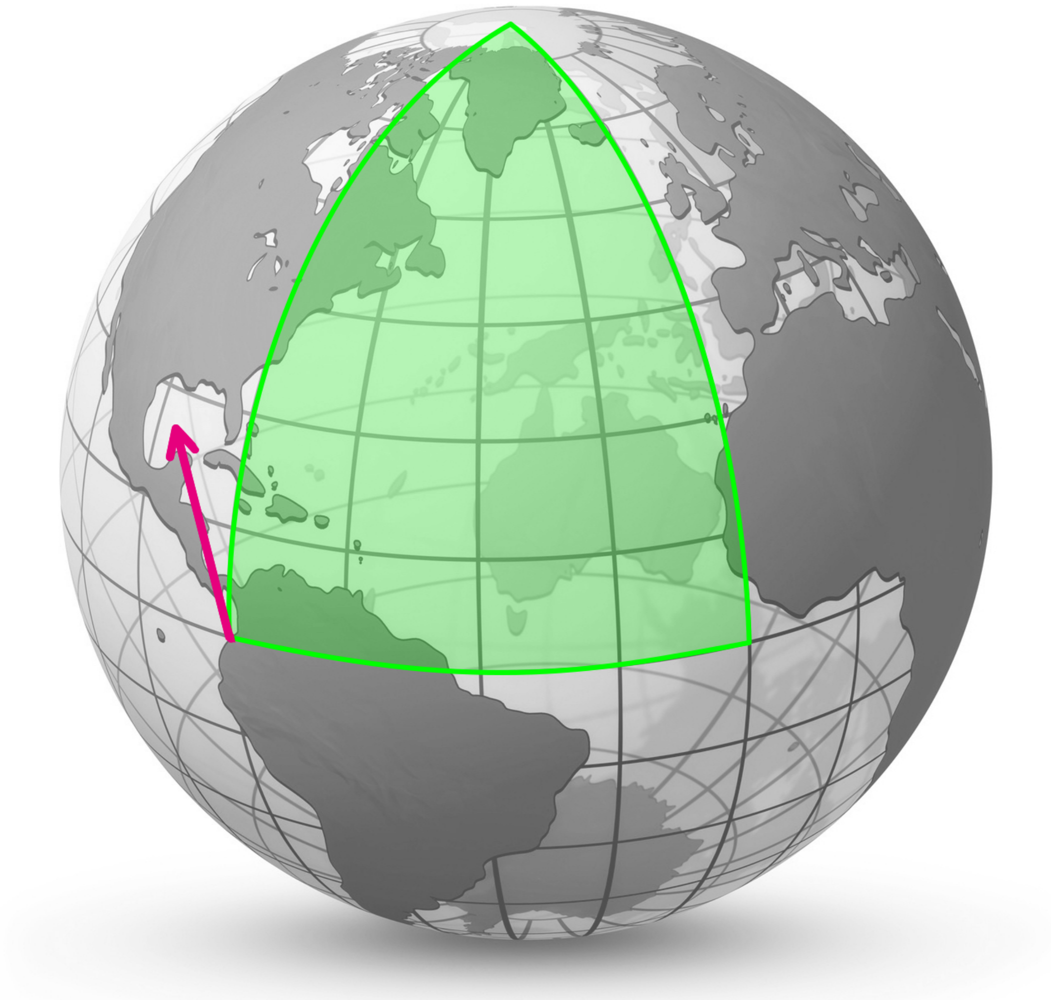

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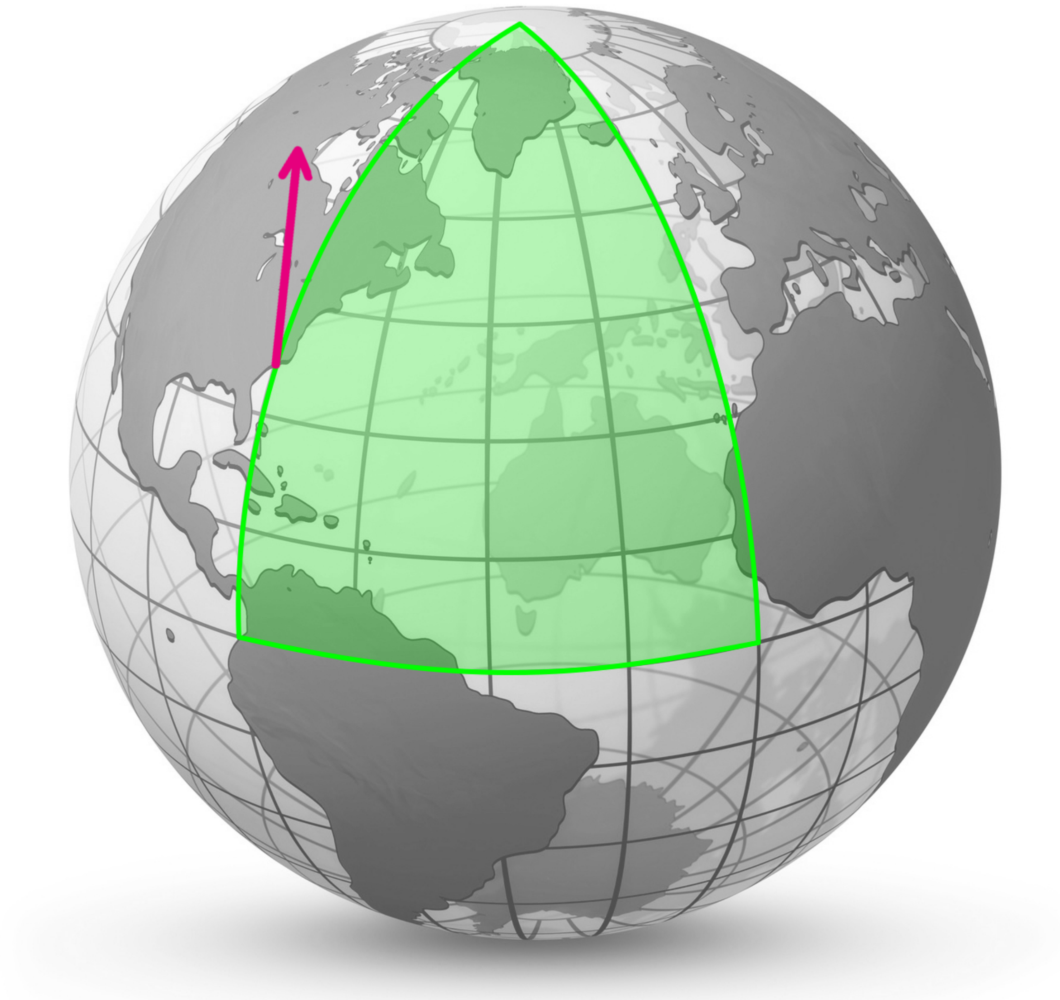

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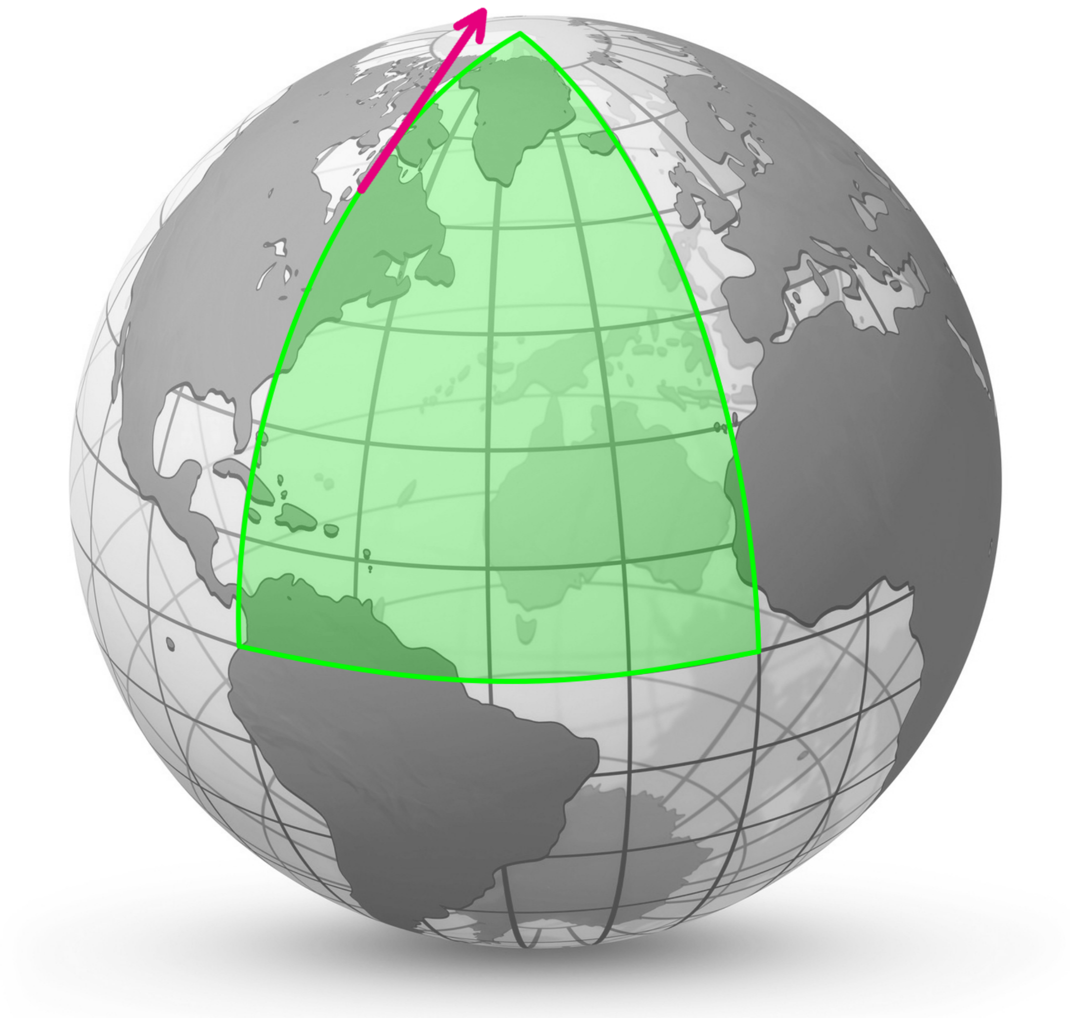

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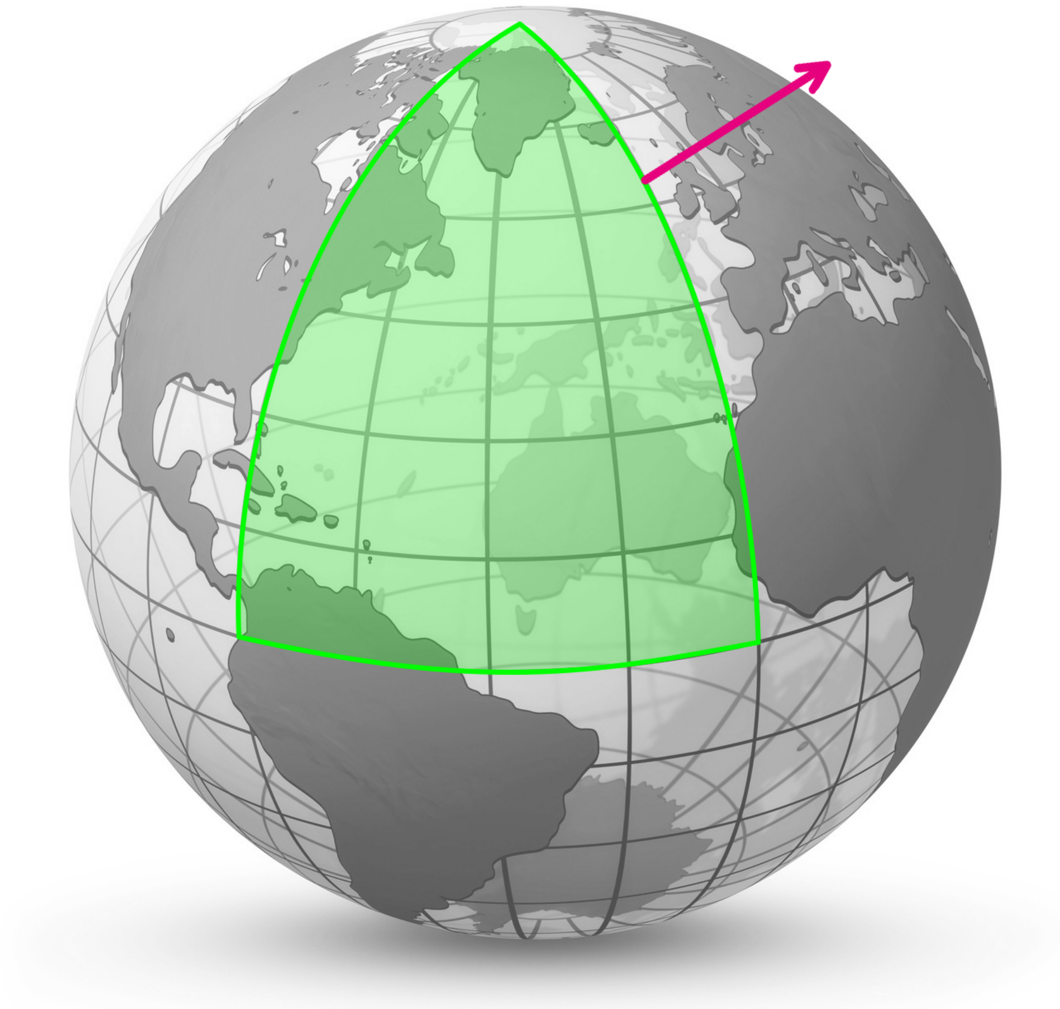

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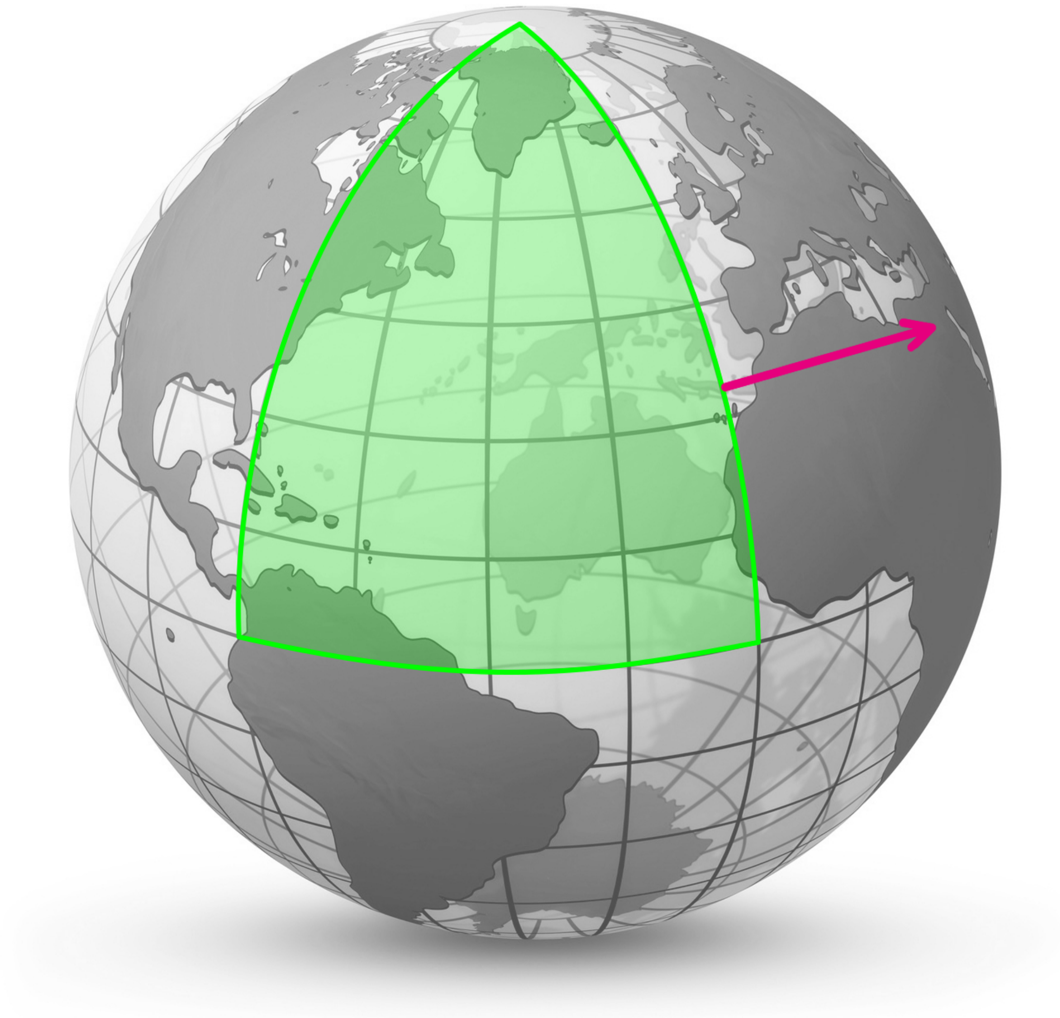

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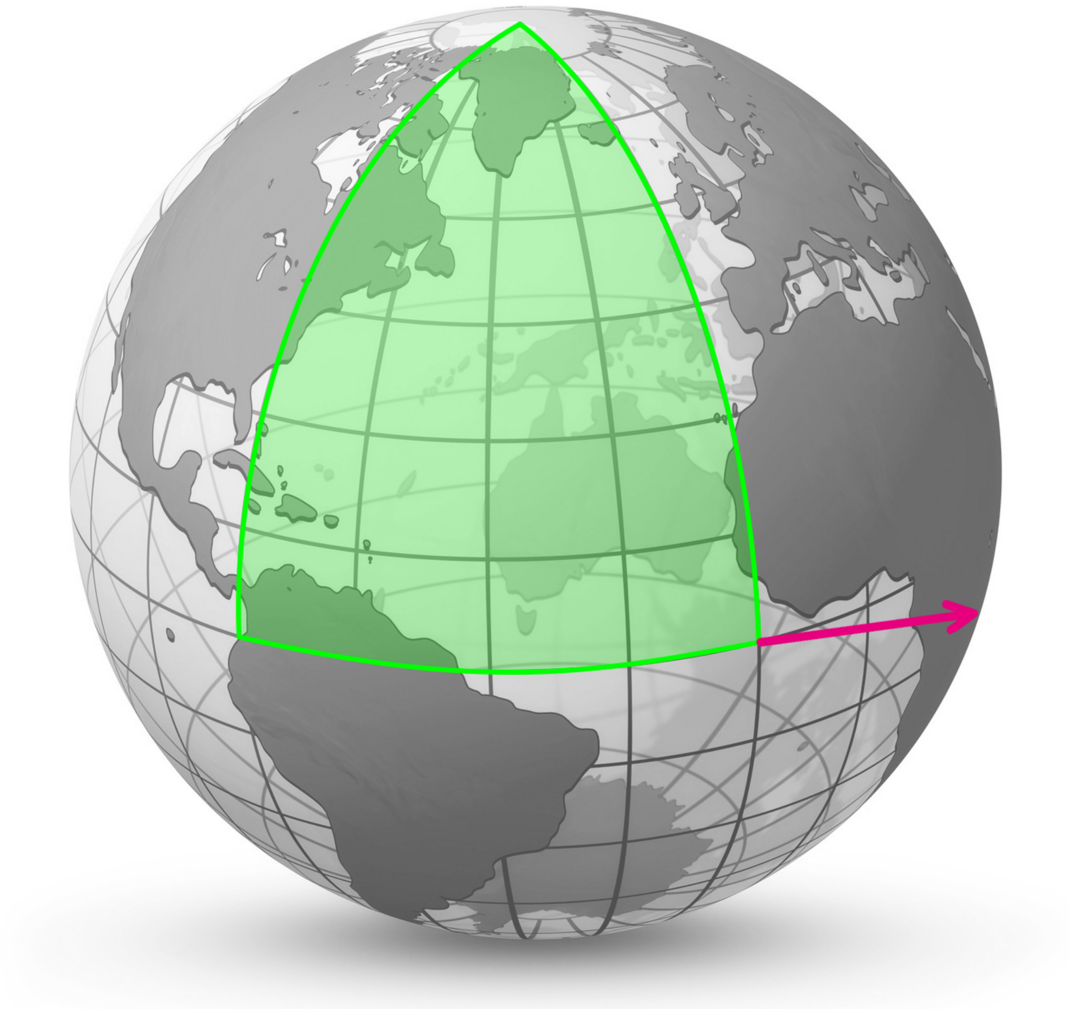

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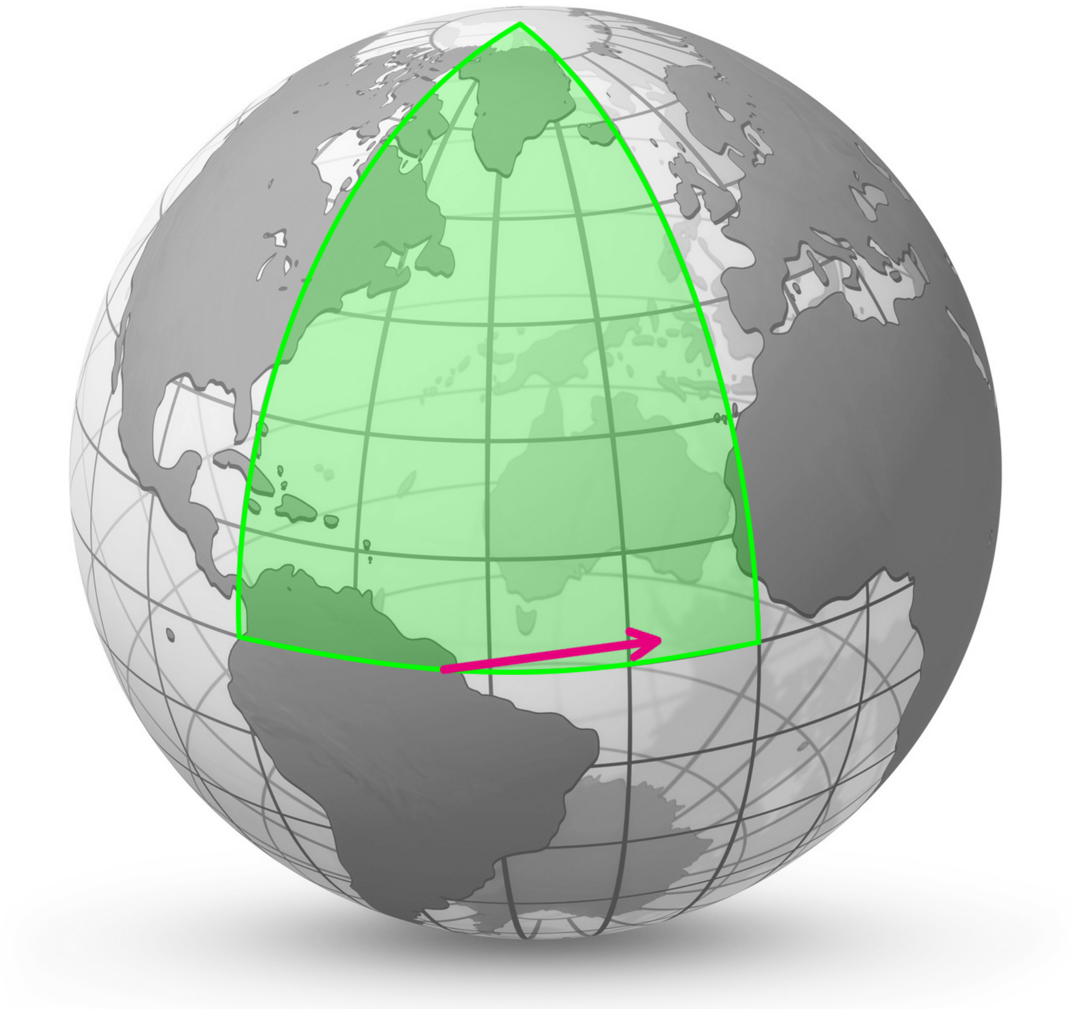

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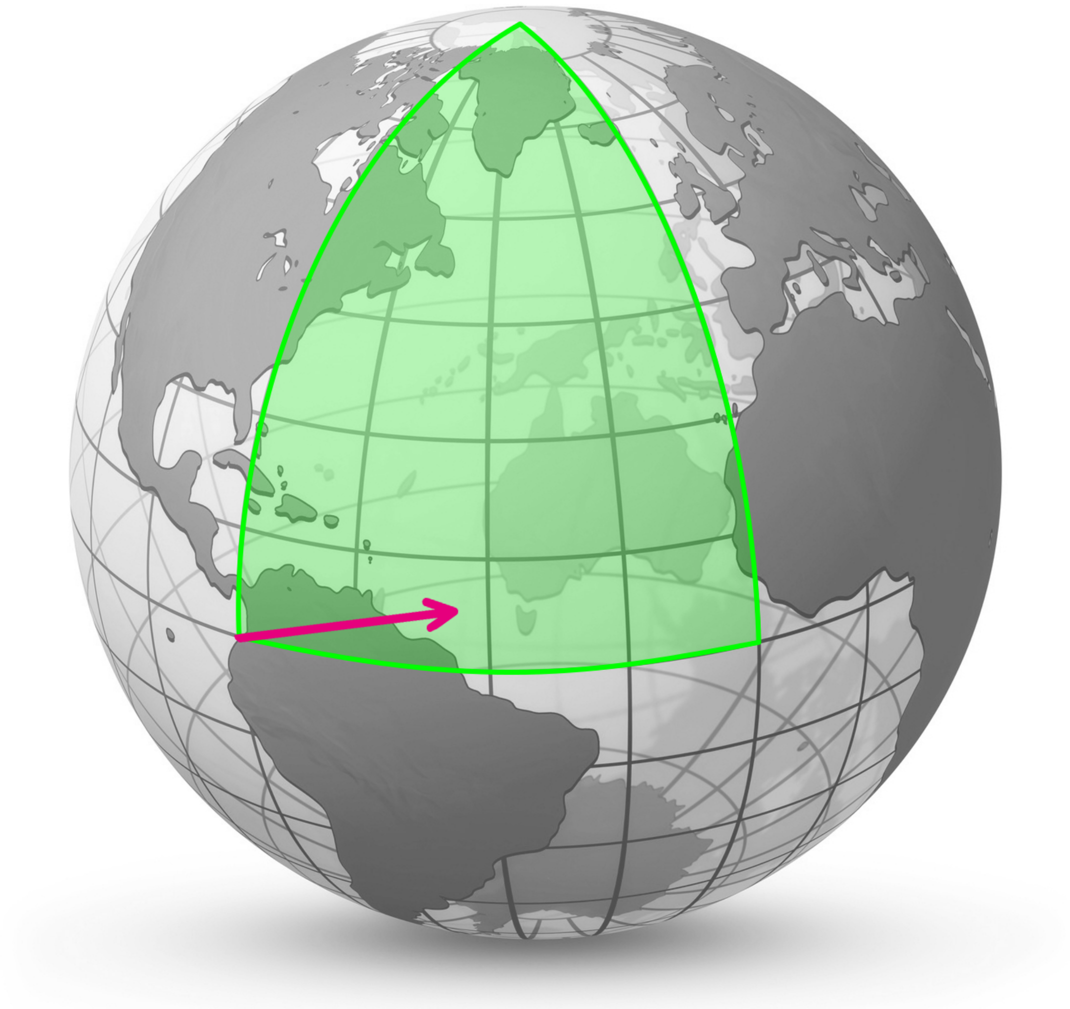

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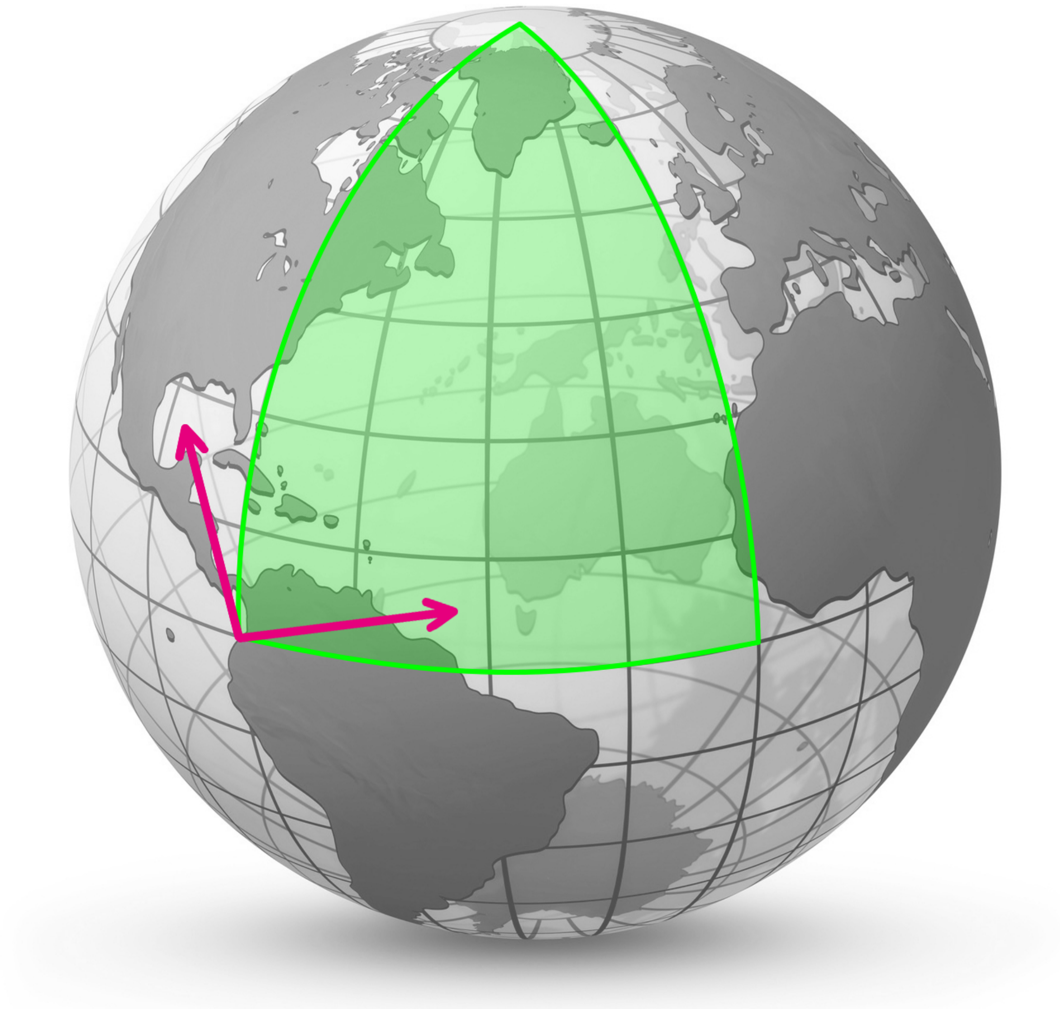

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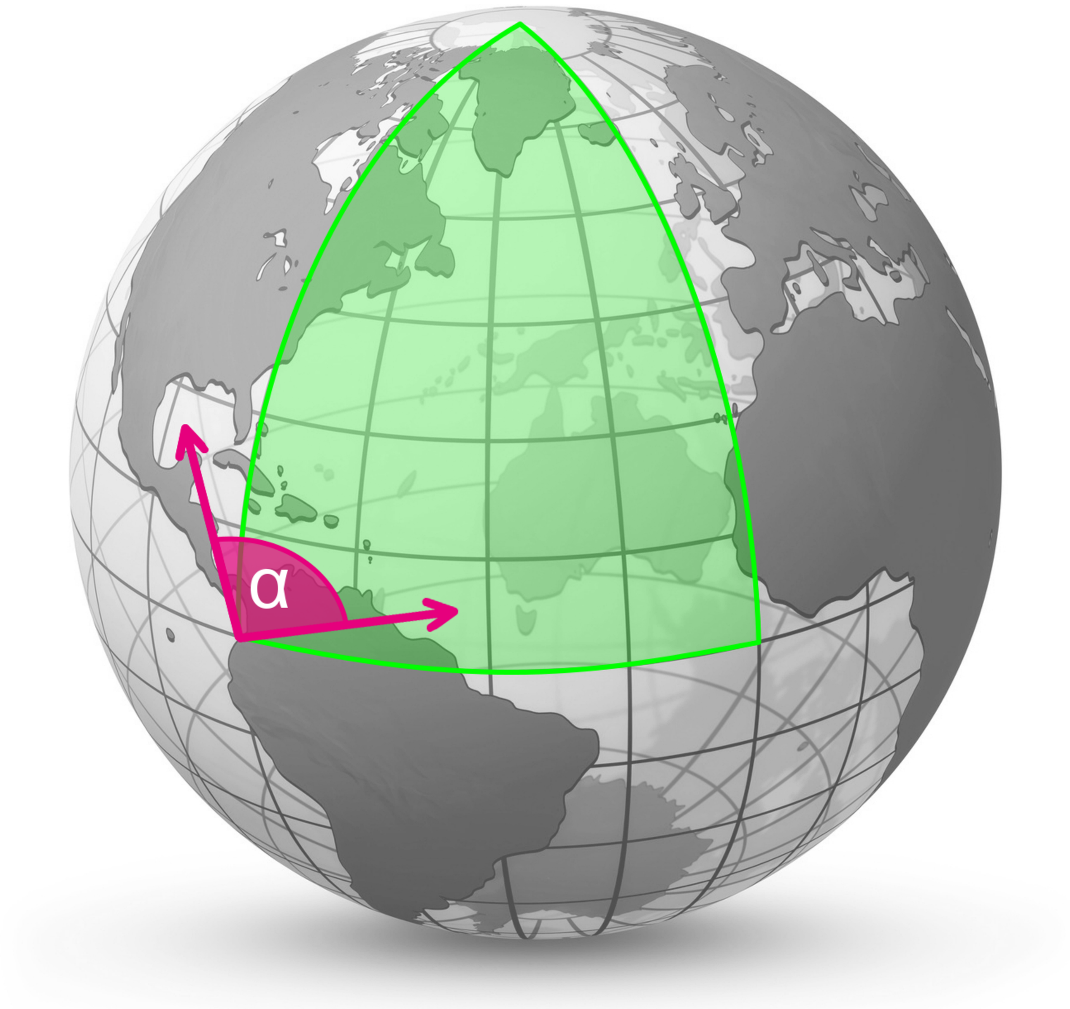
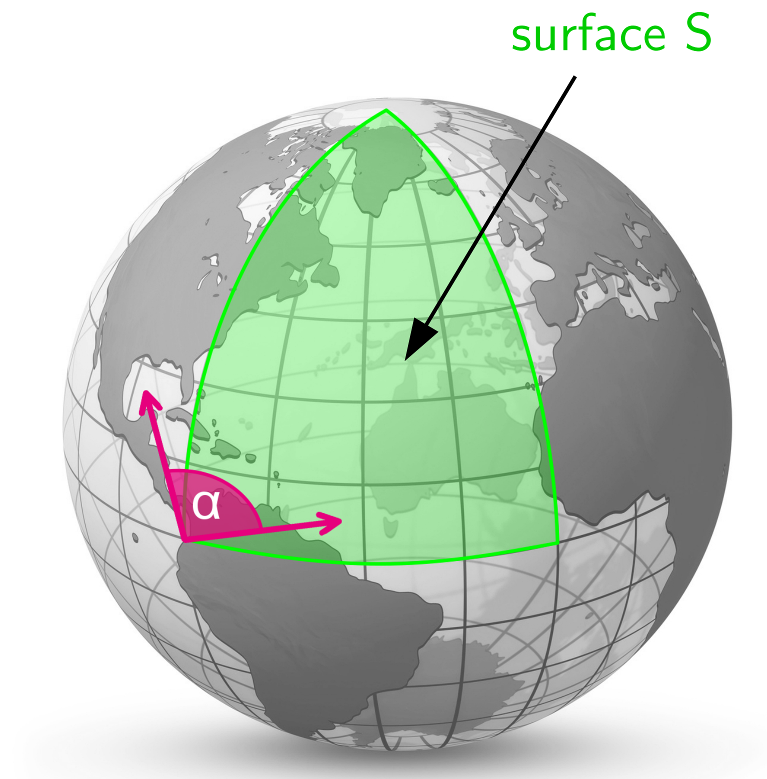


Another example: General Relativity

Three facts:

- On the manifold, we have a non-vanishing curvature 2-form: $R^{\mu}{}_{\nu} = \frac{1}{2} R^{\mu}{}_{\nu\alpha\beta} dx^{\alpha} \wedge dx^{\beta}$
- Curvature 2-form related to the connection 1-form $\Gamma^{\mu}{}_{\nu}$ via $R^{\mu}{}_{\nu} := d\Gamma^{\mu}{}_{\nu} + \Gamma^{\mu}{}_{\alpha} \wedge \Gamma^{\alpha}{}_{\nu}$
- Parallel transport of **vector** along closed loop ∂S yields a rotation:

$$M^{\mu}{}_{\nu} = \int_{S} \frac{1}{2} R^{\mu}{}_{\nu\alpha\beta} dx^{\alpha} \wedge dx^{\beta}$$



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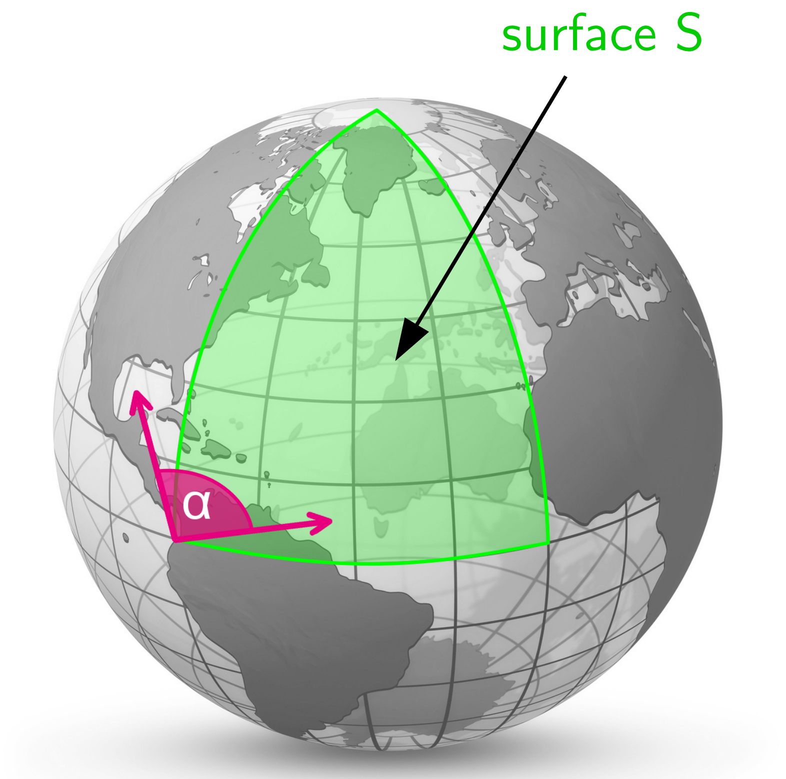


Interpretation: $v^\mu = M^\mu{}_\alpha v^\alpha$

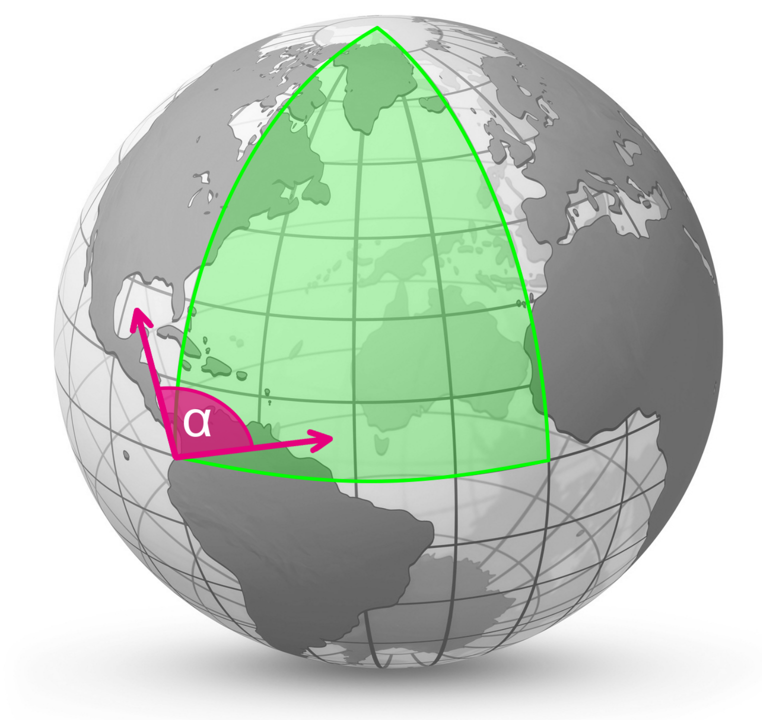
Remember: $R^\mu{}_\nu =$ "$D\Gamma^\mu{}_\nu$" (just like $F = dA$)

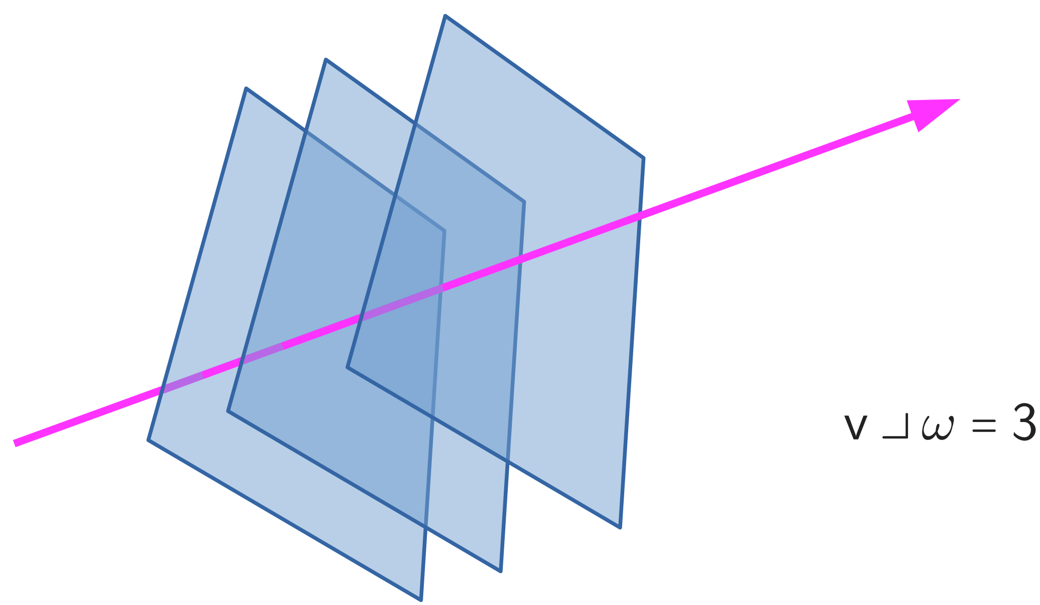
$\rightarrow$ Stokes: The matrix $M^\mu{}_\nu$ is the holonomy of the curvature 2-form around ∂S .

III. Conclusions

We find:

- Differential forms tell us about physics.
- They can be (locally) visualized as planes counting intersections with vectors.
- They have a broad application range, including gauge theories.





Abstract

Differential forms: from classical force to the Wilson loop

We start by reviewing basic properties of differential forms in three dimensions. Using the Hodge star and thereby deriving a visualization procedure, we move on to classical mechanics and vacuum electrodynamics. Therein, differential forms can be interpreted operationally, and their full physical significance becomes clear.

We now move on to more abstract grounds: we revisit electrodynamics as a gauge theory, and discuss its connection 1-form and its relation to the group U(1). We close by motivating the geometric interpretation of connection 1-forms in gauge theories using the Wilson loop, and sketch its application to General Relativity and beyond.