recap:

$$\left.\begin{aligned} \nabla_\rho T^\mu{}_\nu &= \partial_\rho T^\mu{}_\nu + \Gamma^\mu{}_{\rho\alpha} T^\alpha{}_\nu - \Gamma^\alpha{}_{\rho\nu} T^\mu{}_\alpha \\ \nabla_\rho V^\mu &= \partial_\rho V^\mu + \Gamma^\mu{}_{\rho\alpha} V^\alpha \\ \nabla_\rho \omega_\mu &= \partial_\rho \omega_\mu - \Gamma^\alpha{}_{\rho\mu}\omega_\alpha \end{aligned}\right\} \text{transform as tensors under coordinate trafos}$$

"covariant derivative" $\longrightarrow$ ∇_μ instead of ∂_μ = partial derivative

Recall: transforming $x^\mu \to y^{\mu'}$ implies relation between $\Gamma^\mu{}_{\nu\rho}$ and $\Gamma'^{\mu'}{}_{\nu'\rho'}$:

$$\Gamma^\mu{}_{\nu\rho} = \frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial y^{\nu'}}{\partial x^\nu}\frac{\partial y^{\rho'}}{\partial x^\rho}\Gamma'^{\mu'}{}_{\nu'\rho'} + \frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial^2 y^{\mu'}}{\partial x^\nu \partial x^\rho}$$

Let's prove that $\nabla_\rho V^\mu$ and $\nabla_\rho \omega_\mu$ are components of a tensor!

1) $\nabla_\rho V^\mu \equiv \partial_\rho V^\mu + \Gamma^\mu{}_{\rho\alpha}V^\alpha$

$$= \frac{\partial y^{\rho'}}{\partial x^\rho}\frac{\partial}{\partial y^{\rho'}}\frac{\partial x^\mu}{\partial y^{\mu'}}V^{\mu'} + \frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial y^{\rho'}}{\partial x^\rho}\frac{\partial y^{\alpha'}}{\partial x^\alpha}\Gamma'^{\mu'}{}_{\rho'\alpha'}\frac{\partial x^\alpha}{\partial y^{\beta'}}V^{\beta'}$$
$$+ \frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial^2 y^{\mu'}}{\partial x^\rho \partial x^\alpha}\frac{\partial x^\alpha}{\partial y^{\alpha'}}V^{\alpha'}$$

$$= \frac{\partial y^{\rho'}}{\partial x^\rho}\frac{\partial x^\mu}{\partial y^{\mu'}}\left(\partial_{\rho'}V^{\mu'} + \Gamma'^{\mu'}{}_{\rho'\alpha'}V^{\alpha'}\right)$$
$$+ \boxed{\frac{\partial y^{\rho'}}{\partial x^\rho}\frac{\partial^2 x^\mu}{\partial y^{\rho'}\partial y^{\mu'}}}V^{\mu'} + \frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial^2 y^{\mu'}}{\partial x^\rho \partial x^\alpha}\frac{\partial x^\alpha}{\partial y^{\alpha'}}V^{\alpha'}$$

$$= \frac{\partial y^{\rho'}}{\partial x^\rho}\frac{\partial x^\mu}{\partial y^{\mu'}}\nabla_{\rho'}V^{\mu'} \underbrace{- \frac{\partial x^\mu}{\partial y^{\rho'}}\frac{\partial^2 y^{\rho'}}{\partial y^{\mu'}\partial x^\rho}V^{\mu'} + \frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial^2 y^{\mu'}}{\partial x^\rho \partial x^\alpha}\frac{\partial x^\alpha}{\partial y^{\alpha'}}V^{\alpha'}}_{=0\,!}$$

where we used the identity:

$$\frac{\partial}{\partial y^{\mu'}}\left(\frac{\partial x^\mu}{\partial y^{\rho'}}\frac{\partial y^{\rho'}}{\partial x^\rho}\right) = 0 = \frac{\partial^2 x^\mu}{\partial y^{\mu'}\partial y^{\rho'}}\frac{\partial y^{\rho'}}{\partial x^\rho} + \frac{\partial x^\mu}{\partial y^{\rho'}}\frac{\partial^2 y^{\rho'}}{\partial y^{\mu'}\partial x^\rho}$$

2) $\nabla_\rho \omega_\mu \longrightarrow$ homework!

Curvature and torsion:

$$\nabla_\alpha \nabla_\beta v^\mu - \nabla_\beta \nabla_\alpha v^\mu = R_{\alpha\beta}{}^\mu{}_\nu v^\nu - T_{\alpha\beta}{}^\lambda \nabla_\lambda v^\mu ,$$

curvature: $$R_{\alpha\beta}{}^\mu{}_\nu = \partial_\alpha \Gamma^\mu{}_{\beta\nu} - \partial_\beta \Gamma^\mu{}_{\alpha\nu} + \Gamma^\mu{}_{\alpha\lambda}\Gamma^\lambda{}_{\beta\nu} - \Gamma^\mu{}_{\beta\lambda}\Gamma^\lambda{}_{\alpha\nu} ,$$

torsion: $$T_{\alpha\beta}{}^\mu = \Gamma^\mu{}_{\alpha\beta} - \Gamma^\mu{}_{\beta\alpha} .$$

Let's derive that!

$$\nabla_\alpha \nabla_\beta v^\mu \equiv \nabla_\alpha X^\mu{}_\beta = \partial_\alpha X^\mu{}_\beta + \Gamma^\mu{}_{\alpha\lambda} X^\lambda{}_\beta - \Gamma^\lambda{}_{\alpha\beta} X^\mu{}_\lambda$$

$$= \partial_\alpha \left(\partial_\beta v^\mu + \Gamma^\mu{}_{\beta\lambda} v^\lambda \right) + \Gamma^\mu{}_{\alpha\lambda} \left(\partial_\beta v^\lambda + \Gamma^\lambda{}_{\beta\rho} v^\rho \right) - \Gamma^\lambda{}_{\alpha\beta} \left(\partial_\lambda v^\mu + \Gamma^\mu{}_{\lambda\rho} v^\rho \right)$$

$$= \partial_\alpha \partial_\beta v^\mu + \left(\partial_\alpha \Gamma^\mu{}_{\beta\lambda}\right) v^\lambda + \Gamma^\mu{}_{\beta\lambda} \partial_\alpha v^\lambda + \Gamma^\mu{}_{\alpha\lambda} \partial_\beta v^\lambda - \Gamma^\lambda{}_{\alpha\beta} \partial_\lambda v^\mu + \Gamma^\mu{}_{\alpha\lambda} \Gamma^\lambda{}_{\beta\rho} v^\rho - \Gamma^\lambda{}_{\alpha\beta} \Gamma^\mu{}_{\lambda\rho} v^\rho$$

$$\nabla_\beta \nabla_\alpha v^\mu = \left(\nabla_\alpha \nabla_\beta v^\mu\right) \text{ with } \alpha \leftrightarrow \beta \,!$$

$$\rightarrow \left(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha\right) v^\mu = \cancelto{0}{\left(\partial_\alpha \partial_\beta - \partial_\beta \partial_\alpha\right)} v^\mu + \underbrace{\left(\partial_\alpha \Gamma^\mu{}_{\beta\lambda} - \partial_\beta \Gamma^\mu{}_{\alpha\lambda}\right) v^\lambda}$$
$$+ \cancelto{2}{\Gamma^\mu{}_{\beta\lambda} \partial_\alpha v^\lambda} - \cancelto{1}{\Gamma^\mu{}_{\alpha\lambda} \partial_\beta v^\lambda} + \cancelto{1}{\Gamma^\mu{}_{\alpha\lambda} \partial_\beta v^\lambda} - \cancelto{2}{\Gamma^\mu{}_{\beta\lambda} \partial_\alpha v^\lambda}$$
$$- \left(\Gamma^\lambda{}_{\alpha\beta} - \Gamma^\lambda{}_{\beta\alpha}\right) \partial_\lambda v^\mu + \underbrace{\Gamma^\mu{}_{\alpha\lambda} \Gamma^\lambda{}_{\beta\rho} v^\rho - \Gamma^\mu{}_{\beta\lambda} \Gamma^\lambda{}_{\alpha\rho} v^\rho} - \left(\Gamma^\lambda{}_{\alpha\beta} - \Gamma^\lambda{}_{\beta\alpha}\right) \Gamma^\mu{}_{\lambda\rho} v^\rho$$

$$= \left(\partial_\alpha \Gamma^\mu{}_{\beta\rho} - \partial_\beta \Gamma^\mu{}_{\alpha\rho} + \Gamma^\mu{}_{\alpha\lambda} \Gamma^\lambda{}_{\beta\rho} - \Gamma^\mu{}_{\beta\lambda} \Gamma^\lambda{}_{\alpha\rho} \right) v^\rho$$
$$- \left(\Gamma^\lambda{}_{\alpha\beta} - \Gamma^\lambda{}_{\beta\alpha}\right)\left(\partial_\lambda v^\mu + \Gamma^\mu{}_{\lambda\rho} v^\rho \right) = R_{\alpha\beta}{}^\mu{}_\nu v^\nu - T_{\alpha\beta}{}^\lambda \nabla_\lambda v^\mu$$

$\rightarrow$ curvature and torsion measure how exactly covariant derivatives don't commute!

For a scalar: $$\nabla_\alpha \nabla_\beta \phi = \nabla_\alpha \partial_\beta \phi = \partial_\alpha \partial_\beta \phi - \Gamma^\lambda{}_{\alpha\beta} \partial_\lambda \phi$$

$$[\nabla_\alpha, \nabla_\beta]\phi = -T_{\alpha\beta}{}^\lambda \nabla_\lambda \phi$$

Rules for covariant derivative:

i) linearity: $\nabla_\rho(\alpha T_{\mu\nu}) = \alpha \nabla_\rho T_{\mu\nu}$

ii) Leibniz rule: $\nabla_\rho(V^\mu T^\nu{}_\sigma) = (\nabla_\rho V^\mu) T^\nu{}_\sigma + V^\mu(\nabla_\rho T^\nu{}_\sigma)$

iii) scalar case: $\nabla_\rho f = \partial_\rho f$ (reduces to partial derivative when acting on a scalar)

Next time:

$\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ = covariant Laplacian

also: assuming $\nabla_\rho g_{\mu\nu} = 0 \;\wedge\; \Gamma^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\rho\nu}$ (no torsion)

gives a unique connection ("Levi-Civita connection")

$$\tilde{\Gamma}^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}\left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}\right)$$

$\triangleq$ covariant derivative in General Relativity