5. The covariant derivative

Q: How do we differentiate a tensor and get back a tensor?

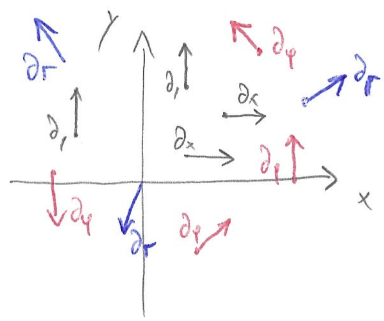
$$\partial_\rho T^\mu{}_\nu \longrightarrow \frac{\partial y^{\rho'}}{\partial x^\rho}\frac{\partial}{\partial y^{\rho'}}\left[\frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial y^{\nu'}}{\partial x^\nu} T^{\mu'}{}_{\nu'}\right]$$

$$= \frac{\partial y^{\rho'}}{\partial x^\rho}\frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial y^{\nu'}}{\partial x^\nu}\partial_{\rho'} T^{\mu'}{}_{\nu'} + \underbrace{\frac{\partial y^{\rho'}}{\partial x^\rho}\left[\frac{\partial^2 x^\mu}{\partial y^{\rho'}\partial y^{\mu'}}\frac{\partial y^{\nu'}}{\partial x^\nu} + \frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial^2 y^{\nu'}}{\partial y^{\rho'}\partial x^\nu}\right] T^{\mu'}{}_{\nu'}}_{\text{non-tensorial behavior.}}$$

→ Naive derivative does not work, does not give a new tensor.

What went wrong?

Recap:



In general, the basis (and cobasis) also depends on x^μ, so we need to take that into account when differentiating tensors.

Connection:

$$\partial_\mu \partial_\nu = \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} = \Gamma^\alpha{}_{\mu\nu}\frac{\partial}{\partial x^\alpha} = \Gamma^\alpha{}_{\mu\nu}\partial_\alpha$$

$$\partial_\mu dx^\nu = \frac{\partial}{\partial x^\mu} dx^\nu = \bar{\Gamma}_{\alpha\mu}{}^\nu dx^\alpha \overset{*}{=} -\Gamma^\nu{}_{\mu\alpha} dx^\alpha$$

The coefficients $\Gamma^\alpha{}_{\mu\nu}$ are called the "connection." The $\bar{\Gamma}_{\alpha\mu}{}^\nu$ are related to them:

Know that $\underbrace{\partial_\rho \delta^\mu_\nu \overset{!}{=} 0}$ $= \partial_\rho \partial_\nu \lrcorner\, dx^\mu = (\partial_\rho \partial_\nu)\lrcorner\, dx^\mu + \partial_\nu \lrcorner (\partial_\rho dx^\mu)$

$$\equiv \Gamma^\alpha{}_{\rho\nu}(\partial_\alpha \lrcorner\, dx^\mu) + \bar{\Gamma}_{\alpha\rho}{}^\mu(\partial_\nu \lrcorner\, dx^\alpha) = \Gamma^\mu{}_{\rho\nu} + \bar{\Gamma}_{\nu\rho}{}^\mu = 0$$

$$\rightarrow \bar{\Gamma}_{\alpha\mu}{}^\nu = -\Gamma^\nu{}_{\mu\alpha} \quad (*)$$

Notation:

$$\Gamma^\alpha{}_{\mu\nu}\partial_\alpha = \partial_\mu \partial_\nu, \qquad -\Gamma^\nu{}_{\mu\alpha} dx^\alpha = \partial_\mu dx^\nu$$

In $\Gamma^\alpha{}_{\mu\nu}$: α — vector index; μ — w.r.t. what component do we differentiate; ν — which basis vector is differentiated.

In $-\Gamma^\nu{}_{\mu\alpha}$: ν — which cobasis covector is differentiated; μ — w.r.t. what component do we differentiate; α — covector index.

Now we can compute: Oct 8, 2021

$$\partial_\rho \underline{T} = \partial_\rho T^\mu{}_\nu \, \partial_\mu \otimes dx^\nu$$

$$= (\partial_\rho T^\mu{}_\nu)\, \partial_\mu \otimes dx^\nu + T^\mu{}_\nu (\partial_\rho \partial_\mu) \otimes dx^\nu + T^\mu{}_\nu \, \partial_\mu \otimes (\partial_\rho dx^\nu)$$

$$= \partial_\rho T^\mu{}_\nu \, \partial_\mu \otimes dx^\nu + T^\mu{}_\nu \Gamma^\alpha{}_{\rho\mu} \partial_\alpha \otimes dx^\nu - T^\mu{}_\nu \Gamma^\nu{}_{\rho\alpha} \partial_\mu \otimes dx^\alpha$$

$$= \underbrace{\left[\partial_\rho T^\mu{}_\nu + \Gamma^\mu{}_{\rho\alpha} T^\alpha{}_\nu - \Gamma^\alpha{}_{\rho\nu} T^\mu{}_\alpha\right]}_{\substack{\text{tensorial components } \nabla_\rho T^\mu{}_\nu \\ \text{"covariant derivative"}}} \partial_\mu \otimes dx^\nu$$

How does $\Gamma^\alpha{}_{\mu\nu}$ transform under coordinate transformations?

$$\partial_\mu \partial_\nu = \Gamma^\alpha{}_{\mu\nu} \partial_\alpha = \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} = \frac{\partial y^{\mu'}}{\partial x^\mu} \frac{\partial}{\partial y^{\mu'}} \left[\frac{\partial y^{\nu'}}{\partial x^\nu} \frac{\partial}{\partial y^{\nu'}} \right]$$

$$= \frac{\partial y^{\mu'}}{\partial x^\mu} \frac{\partial y^{\nu'}}{\partial x^\nu} \underbrace{\frac{\partial}{\partial y^{\mu'}} \frac{\partial}{\partial y^{\nu'}}}_{\equiv} + \frac{\partial y^{\mu'}}{\partial x^\mu} \frac{\partial^2 y^{\nu'}}{\partial y^{\mu'} \partial x^\nu} \frac{\partial}{\partial y^{\nu'}}$$

$$= \frac{\partial y^{\mu'}}{\partial x^\mu} \frac{\partial y^{\nu'}}{\partial x^\nu} \Gamma'^{\alpha'}{}_{\mu'\nu'} \frac{\partial}{\partial y^{\alpha'}} + \frac{\partial^2 y^{\alpha'}}{\partial x^\mu \partial x^\nu} \frac{\partial}{\partial y^{\alpha'}}$$

$$\Longrightarrow \Gamma^\alpha{}_{\mu\nu} = \frac{\partial x^\alpha}{\partial y^{\alpha'}} \left[\underbrace{\frac{\partial y^{\mu'}}{\partial x^\mu} \frac{\partial y^{\nu'}}{\partial x^\nu} \Gamma'^{\alpha'}{}_{\mu'\nu'}}_{\text{like a tensor}} + \underbrace{\frac{\partial^2 y^{\alpha'}}{\partial x^\mu \partial x^\nu}}_{\text{but extra stuff!}} \right]$$

<u>Important</u>: $\Gamma^\alpha{}_{\mu\nu}$ is <u>not</u> a tensor.

It transforms inhomogeneously!