3. Coordinates + coordinate transformations

<u>Recap:</u>

$$\underline{T} = \underbrace{T^i{}_j \, \hat{e}_i \otimes \hat{\vartheta}^j}_{\text{abstract basis}} = \underbrace{T^\mu{}_\nu \, \frac{\partial}{\partial x^\mu} \otimes dx^\nu}_{\text{coordinate basis}}$$

$$\left[\begin{array}{ll} \hat{e}_i = e^\mu{}_i \dfrac{\partial}{\partial x^\mu}, & \hat{\vartheta}^j = e_\nu{}^j \, dx^\nu \\ \Longleftrightarrow \dfrac{\partial}{\partial x^\mu} = e_\mu{}^i \, \hat{e}_i, & dx^\nu = e^\nu{}_j \, \hat{\vartheta}^j \end{array}\right\} \text{relation between bases}$$

$$T^\mu{}_\nu = e^\mu{}_i \, e_\nu{}^j \, T^i{}_j \quad \Longleftrightarrow \quad T^i{}_j = e_\mu{}^i \, e^\nu{}_j \, T^\mu{}_\nu$$

relation between tensor components

Basis and cobasis are always dual: $\hat{e}_i \lrcorner \hat{\vartheta}^j = \delta^j_i$ and $\frac{\partial}{\partial x^\mu} \lrcorner dx^\nu = \delta^\nu_\mu$

This implies $e^\mu{}_i \, e_\mu{}^j = \delta^j_i$ and $e_\mu{}^i \, e^\nu{}_i = \delta^\nu_\mu$.

<u>Now:</u> let us focus a bit on coordinates!

<u>Notation:</u> $x^\mu = (x, y, z)$ or $x^\mu = (r, \theta, \varphi)$ μ numbers the coordinates.

$x^x = x, \; x^y = y, \; x^z = z$ $\quad$ $x^r = r, \; x^\theta = \theta, \; x^\varphi = \varphi$ (looks a bit confusing)

e.g. $\underline{\eta} = 1 \, dx \otimes dx + dy \otimes dy + dz \otimes dz = \eta_{\mu\nu} \, dx^\mu \otimes dx^\nu$

this 1 here is η_{xx}.

e.g. $\underline{v} = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \equiv x \partial_y - y \partial_x = v^\mu \frac{\partial}{\partial x^\mu} \equiv v^\mu \partial_\mu$

$= v^x \partial_x + v^y \partial_y \longrightarrow v^x = -y, \; v^y = x$

e.g. $\underline{M} = 24z \, \partial_x \otimes dz = M^\mu{}_\nu \, \partial_\mu \otimes dx^\nu$

$\partial_\mu \otimes dx^\nu$

$\mu = x \quad \nu = z \longrightarrow M^x{}_z = 24z$.

Coordinate transformations:

$$x^\mu = (x, y, z) \longrightarrow y^{\nu'} = (r, \theta, \varphi)$$

$\mu = x, y, z$ new symbol new index $\nu' = r, \theta, \varphi$

Know how coordinate (co-) basis transforms; chain rule!

$$\frac{\partial}{\partial x^\mu} = \frac{\partial y^{\nu'}}{\partial x^\mu} \frac{\partial}{\partial y^{\nu'}} , \qquad dx^\mu = \frac{\partial x^\mu}{\partial y^{\nu'}} dy^{\nu'}$$

What about tensors? E.g. $\binom{1}{1}$ tensor $\underline{T}$?

Must have: $\underline{T} = T^\mu{}_\nu \frac{\partial}{\partial x^\mu} \otimes dx^\nu = T^{\mu'}{}_{\nu'} \frac{\partial}{\partial y^{\mu'}} \otimes dy^{\nu'}$

Remember: a tensor $\underline{T}$ is a geometric object that can be described by any basis!

Question: what's the relation between $T^\mu{}_\nu$ and $T^{\mu'}{}_{\nu'}$?

$$T^{\mu'}{}_{\nu'} \frac{\partial}{\partial y^{\mu'}} \otimes dy^{\nu'} = T^{\mu'}{}_{\nu'} \frac{\partial x^\mu}{\partial y^{\mu'}} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial y^{\nu'}}{\partial x^\nu} dx^\nu$$

$$= \underbrace{T^{\mu'}{}_{\nu'} \frac{\partial x^\mu}{\partial y^{\mu'}} \frac{\partial y^{\nu'}}{\partial x^\nu}}_{\equiv T^\mu{}_\nu} \frac{\partial}{\partial x^\mu} \otimes dx^\nu$$

$$\left.\begin{aligned} &\rightarrow T^\mu{}_\nu = \frac{\partial x^\mu}{\partial y^{\mu'}} \frac{\partial y^{\nu'}}{\partial x^\nu} T^{\mu'}{}_{\nu'} \\ &\rightarrow T^{\mu'}{}_{\nu'} = \frac{\partial y^{\mu'}}{\partial x^\mu} \frac{\partial x^\nu}{\partial y^{\nu'}} T^\mu{}_\nu \end{aligned}\right\} \quad \text{use: } \frac{\partial x^\mu}{\partial y^{\mu'}} \frac{\partial y^{\mu'}}{\partial x^\nu} = \delta^\mu_\nu$$

Quick + easy way to compute tensors in a new basis: forget about indices!

Transforming $\underline{v} = x\partial_y - y\partial_x$ to (ρ, φ)-coordinates is easier the following way:

$x = \rho\cos\varphi$ $\quad \frac{\partial\rho}{\partial x} = \cos\varphi \quad \frac{\partial\varphi}{\partial x} = \frac{-\sin\varphi}{\rho}$

$y = \rho\sin\varphi$ $\quad \frac{\partial\rho}{\partial y} = \sin\varphi \quad \frac{\partial\varphi}{\partial y} = \frac{\cos\varphi}{\rho}$

$$\longrightarrow \quad x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = x\left(\frac{\partial\rho}{\partial y}\frac{\partial}{\partial\rho} + \frac{\partial\varphi}{\partial y}\frac{\partial}{\partial\varphi}\right) - y\left(\frac{\partial\rho}{\partial x}\frac{\partial}{\partial\rho} + \frac{\partial\varphi}{\partial x}\frac{\partial}{\partial\varphi}\right)$$

$$= \ldots = \partial/\partial\varphi = \partial_\varphi .$$