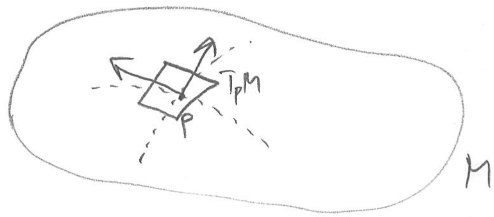
11. Geometry of 2D surfaces

Let us connect differential geometry with simple 3D Euclidean vector language.

Surface:



$\vec{\Psi} = \vec{\Psi}(x^\mu)$, $\mu = 1, \dots, n$

$\vec{\Psi} \in \mathbb{R}^d$ embedding space

Example: $\vec{\Psi}(\theta, \varphi) = r\,(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta) \in \mathbb{R}^3$, $x^\mu = (\theta, \varphi)$.

Basis vectors: $\frac{\partial}{\partial x^\mu} \equiv \frac{\partial \vec{\Psi}}{\partial x^\mu}$ = vectors in $\mathbb{R}^d$.

$\partial_\mu \partial_\nu = \Gamma^\alpha{}_{\mu\nu}\, \partial_\alpha$

General vectors: $\underline{v} = v^\mu \frac{\partial}{\partial x^\mu} \equiv v^\mu \frac{\partial \vec{\Psi}}{\partial x^\mu}$.

Derivative of a vector: $$\partial_\rho \underline{v} = \underbrace{(\partial_\rho v^\mu) \frac{\partial \vec{\Psi}}{\partial x^\mu}}_{\text{OK, } \in T_pM} + v^\mu \underbrace{\frac{\partial^2 \vec{\Psi}}{\partial x^\rho \partial x^\mu}}_{\text{not clear if } \in T_pM}$$

Decompose now: $$\frac{\partial^2 \vec{\Psi}}{\partial x^\mu \partial x^\nu} = \underbrace{\tilde{\Gamma}^\rho{}_{\mu\nu} \frac{\partial \vec{\Psi}}{\partial x^\rho}}_{\substack{\text{tangential part} \\ \checkmark}} + \underbrace{\vec{n}_{\mu\nu}}_{\substack{\text{normal part} \\ \lightning}}$$

$$\rightarrow \quad \partial_\rho \underline{v} = (\partial_\rho v^\mu) \frac{\partial \vec{\Psi}}{\partial x^\mu} + v^\mu \left[\tilde{\Gamma}^\alpha{}_{\rho\mu} \frac{\partial \vec{\Psi}}{\partial x^\alpha} + \vec{n}_{\rho\mu} \right]$$

$$\rightarrow \quad \underbrace{\partial_\rho \underline{v}}_{\substack{\text{general} \\ \text{derivative}}} = \underbrace{\left(\partial_\rho v^\mu + \tilde{\Gamma}^\mu{}_{\rho\alpha} v^\alpha\right) \frac{\partial \vec{\Psi}}{\partial x^\mu}}_{\substack{\text{covariant derivative} \\ \text{with geometrical meaning}}} + \underbrace{\vec{n}_{\rho\mu} v^\mu}_{\substack{\text{non-geometrical} \\ \text{stuff}}}$$

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Let us "derive" the $\Gamma^\mu{}_{\mu\nu}$ coefficients!

1) $$\frac{\partial^2 \vec{\psi}}{\partial x^\mu \partial x^\nu} = \tilde{\Gamma}^\rho{}_{\mu\nu} \frac{\partial \vec{\psi}}{\partial x^\rho} + \vec{n}_{\mu\nu} \quad \Big| \cdot \frac{\partial \vec{\psi}}{\partial x^\sigma}$$

$$\left\langle \frac{\partial^2 \vec{\psi}}{\partial x^\mu \partial x^\nu}, \frac{\partial \vec{\psi}}{\partial x^\sigma} \right\rangle = \tilde{\Gamma}^\rho{}_{\mu\nu} \underbrace{\left\langle \frac{\partial \vec{\psi}}{\partial x^\rho}, \frac{\partial \vec{\psi}}{\partial x^\sigma} \right\rangle}_{\equiv g_{\rho\sigma}} + \underbrace{\left\langle \vec{n}_{\mu\nu}, \frac{\partial \vec{\psi}}{\partial x^\sigma} \right\rangle}_{=0}$$

$$= \tilde{\Gamma}^\rho{}_{\mu\nu} g_{\rho\sigma} \equiv \tilde{\Gamma}_{\sigma\mu\nu}$$

2) Notice that $$\frac{\partial g_{\mu\nu}}{\partial x^\rho} \equiv \frac{\partial}{\partial x^\rho} \left\langle \frac{\partial \vec{\psi}}{\partial x^\mu}, \frac{\partial \vec{\psi}}{\partial x^\nu} \right\rangle$$

$$= \left\langle \frac{\partial^2 \vec{\psi}}{\partial x^\rho \partial x^\mu}, \frac{\partial \vec{\psi}}{\partial x^\nu} \right\rangle + \left\langle \frac{\partial \vec{\psi}}{\partial x^\mu}, \frac{\partial^2 \vec{\psi}}{\partial x^\rho \partial x^\nu} \right\rangle$$

$$\longrightarrow \tilde{\Gamma}_{\mu\nu\rho} = \frac{1}{2} \left(\partial_\nu g_{\mu\rho} + \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} \right)$$

$$\longrightarrow \Gamma^\mu{}_{\nu\rho} = \frac{1}{2} g^{\mu\alpha} \left(\partial_\nu g_{\alpha\rho} + \partial_\rho g_{\alpha\nu} - \partial_\alpha g_{\rho\nu} \right)$$

And now we can compute the curvature of this surface with usual techniques directly from $\tilde{R}_{\alpha\beta}{}^\mu{}_\nu \equiv \partial_\alpha \tilde{\Gamma}^\mu{}_{\beta\nu} - \partial_\beta \tilde{\Gamma}^\mu{}_{\alpha\nu} + \tilde{\Gamma}^\mu{}_{\alpha\lambda} \tilde{\Gamma}^\lambda{}_{\beta\nu} - \tilde{\Gamma}^\mu{}_{\beta\lambda} \tilde{\Gamma}^\lambda{}_{\alpha\nu}$,

$$\tilde{R}_{\mu\nu} \equiv \tilde{R}_{\alpha\mu}{}^\alpha{}_\nu, \quad \tilde{R} = \tilde{R}_{\mu\nu} g^{\mu\nu}.$$