9. Hodge duality

last time: $\underline{\omega} = \omega_{\mu_1 \dots \mu_p} dx^{\mu_1} \otimes \dots \otimes dx^{\mu_p} = \omega_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$

$\underline{\omega}$ p-form $\Leftrightarrow$ ω is a $\binom{0}{p}$ tensor + completely antisymmetric

$\underline{\lambda}$ q-form

$$\underline{\omega} \wedge \underline{\lambda} = (-1)^{p \cdot q} \underline{\lambda} \wedge \underline{\omega}$$

$$d\underline{\omega} \equiv \partial_{[\rho} \omega_{\mu_1 \dots \mu_p]} dx^\rho \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$$

$$dd\underline{\omega} = \partial_{[\sigma} \partial_\rho \omega_{\mu_1 \dots \mu_p]} dx^\sigma \wedge dx^\rho \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} = 0$$

$$d(\underline{\omega} \wedge \underline{\lambda}) = (d\underline{\omega}) \wedge \underline{\lambda} + (-1)^p \underline{\omega} \wedge (d\underline{\lambda}) .$$

recall: p-forms in n dimensions form vector space Λ^p.

$$\dim \Lambda^p = \binom{n}{p} = \frac{n!}{p!(n-p)!} = \dim \Lambda^{n-p}$$

$n = 4$: 1 4 6 4 1

$n = 3$: 1 3 3 1

Hodge dual: isomorphism between $\Lambda^p \leftrightarrow \Lambda^{n-p}$

maps a p-form into an (n-p)-form

need: orientable manifold + ϵ-tensor $\varepsilon_{\mu_1 \mu_2 \dots \mu_n}$

Def:
$$*\underline{\omega} \equiv \frac{1}{p!(n-p)!} \omega^{\mu_1 \mu_2 \dots \mu_p} \varepsilon_{\mu_1 \mu_2 \dots \mu_p \mu_{p+1} \dots \mu_n} dx^{\mu_{p+1}} \wedge \dots \wedge dx^{\mu_n}$$

need a metric to raise indices

$\omega_{\mu_1 \mu_2 \dots \mu_p} \to \omega^{\mu_1 \mu_2 \dots \mu_p}$

$= \underbrace{\sqrt{-g}}_{\sqrt{|\det g_{\mu\nu}|}} \, \underbrace{\epsilon_{\mu_1 \mu_2 \dots \mu_n}}_{\text{totally antisymmetric symbol}}$

$$** \underline{\omega} = (-1)^{p(n-p)+1} \underline{\omega} ,$$ +1 in spacetime from Lorentz signature $(-1, 1, 1, \dots, 1)$

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It all works because of the ε-tensor $\varepsilon_{\underbrace{\mu_1 \cdots \mu_p}_{p} \underbrace{\mu_{p+1} \cdots \mu_n}_{n-p}}$.

Useful identity: $\varepsilon_{\mu_1 \cdots \mu_p \nu_1 \cdots \nu_{n-p}} \varepsilon^{\mu_1 \cdots \mu_p \rho_1 \cdots \rho_{n-p}} = \underset{\substack{\uparrow \\ \text{from Lorentzian} \\ \text{signature}}}{(-1)}\, p! \, \Big(\underbrace{\delta^{\rho_1}_{\nu_1} \delta^{\rho_2}_{\nu_2} \cdots \delta^{\rho_{n-p}}_{\nu_{n-p}} - \delta^{\rho_2}_{\nu_1} \delta^{\rho_1}_{\nu_2} \cdots \delta^{\rho_{n-p}}_{\nu_{n-p}} + \cdots}_{\text{all permutations}} \Big)$

Application of Hodge dual: norm of p-forms.

$$|\underline{\omega}|^2 \equiv \underline{\omega} \wedge * \underline{\omega}$$

$$= \omega_{\mu_1 \cdots \mu_p} \underbrace{dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}} \wedge \frac{1}{p!} \frac{1}{(n-p)!} \omega^{\nu_1 \cdots \nu_p} \varepsilon_{\nu_1 \cdots \nu_p \nu_{p+1} \cdots \nu_n} \underbrace{dx^{\nu_{p+1}} \wedge \cdots \wedge dx^{\nu_n}}$$

$$= \frac{1}{p!} \omega_{\mu_1 \cdots \mu_p} \omega^{\nu_1 \cdots \nu_p} \frac{1}{(n-p)!} \underbrace{\varepsilon_{\nu_1 \cdots \nu_n} \varepsilon^{\mu_1 \cdots \mu_p \nu_{p+1} \cdots \nu_n}}_{(-1)(n-p)!\left(\delta^{\mu_1}_{\nu_1} \delta^{\mu_2}_{\nu_2} \cdots \delta^{\mu_p}_{\nu_p} \pm \text{perm.}\right)} (-\underline{\varepsilon})$$

$\Big(\underline{\varepsilon} = (-1) * 1 = -\frac{1}{n!} \varepsilon_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}$

$\overset{\text{e.g.}}{=} -dt \wedge dx \wedge dy \wedge dz$,

or $=$ $dx \wedge dy \wedge dz$, and so on $\Big)$

"volume form"

$$= \frac{1}{p!} \omega_{\mu_1 \mu_2 \cdots \mu_p} \omega^{\mu_1 \mu_2 \cdots \mu_p} \underline{\varepsilon}$$

Examples: $n = 3$ Euclidean space

$$\underline{\omega} = \omega_x dx + \omega_y dy + \omega_z dz = \omega_\mu dx^\mu$$

$$* \underline{\omega} = \frac{1}{1!} \frac{1}{2!} \omega^\mu \varepsilon_{\mu\nu\rho} dx^\nu \wedge dx^\rho$$

$$= \frac{1}{2} \Big(\omega^x \varepsilon_{xyz} dy \wedge dz + \omega^x \varepsilon_{xzy} dz \wedge dy + \omega^y \varepsilon_{yxz} dx \wedge dz + \omega^y \varepsilon_{yzx} dz \wedge dx + \omega^z \varepsilon_{zxy} dx \wedge dy + \omega^z \varepsilon_{zyx} dy \wedge dx \Big)$$

$\varepsilon_{xyz} = +1$

$$= \omega^x dy \wedge dz + \omega^y dz \wedge dx + \omega^z dx \wedge dy$$

$$\underline{\omega} \wedge * \underline{\omega} = (\omega_x \omega^x + \omega_y \omega^y + \omega_z \omega^z) dx \wedge dy \wedge dz$$

$$= \frac{1}{1!} \omega_\mu \omega^\mu \underline{\varepsilon}, \quad \text{with } \underline{\varepsilon} = dx \wedge dy \wedge dz$$

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Application of Hodge dual: co-derivative

$\delta \equiv *d*$ maps p-forms into (p-1)-forms

Can show (try to prove it): $\delta \underline{\omega} = \partial^{\rho} \omega_{\rho \mu_1 \mu_2 \dots \mu_{p-1}} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_{p-1}}$

Example: $n = 3$ Euclidean space

$\underline{\omega}$ = 1-form

$$d(*\underline{\omega}) = (\partial_x \omega^x)\, dx \wedge dy \wedge dz + (\partial_y \omega^y)\, dy \wedge dz \wedge dx + (\partial_z \omega^z)\, dz \wedge dx \wedge dy$$

$$= (\partial_x \omega^x + \partial_y \omega^y + \partial_z \omega^z)\, dx \wedge dy \wedge dz = (\partial_\mu \omega^\mu)\, \underline{\varepsilon}$$

$$*d(*\underline{\omega}) = (\partial_\mu \omega^\mu) * \underline{\varepsilon} = (\partial_\mu \omega^\mu)(**1) = \partial_\mu \omega^\mu \quad \checkmark$$