Issued: October 8, 2021
Due: 11am, October 15, 2021
Official website: http://spintwo.net/Courses/PHYS-581-Differential-Geometry-for-Physicists/

Please work on this problem set on your own; it should be possible to complete it with the lecture notes and no other external help. If you have questions you can email the instructor, Jens Boos (jboos@wm.edu), or make use of the office hours on Monday, 10am–11am, Small 235. *After* you have completed the assignment please feel free to discuss it with other students.

1 Quick questions

(a) Select the equation(s) with mistakes in them.

[ ] $\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu{}_{\mu\alpha} v^\alpha$
[ ] $\nabla_\mu v^\nu = \partial_\mu v^\nu - \Gamma^\nu{}_{\alpha\mu} v^\alpha$
[ ] $\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu_{\mu\alpha} v^\alpha$
[ ] $\nabla_\mu v^\nu = \partial_\mu v^\nu - \Gamma^\nu_{\alpha\mu} v^\alpha$

(b) Select the equation(s) with mistakes in them.

[ ] $\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu + \Gamma^\alpha_{\nu\mu} \omega_\alpha$
[ ] $\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu + \Gamma^\alpha{}_{\nu\mu} \omega_\alpha$
[ ] $\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\alpha{}_{\mu\nu} \omega_\alpha$
[ ] $\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\alpha_{\mu\nu} \omega_\alpha$

(c) In your own words, why is the covariant derivative necessary to differentiate tensors?

(d) Which one is the correct definition of the connection?

[ ] $\partial_\mu \partial_\nu = \Gamma^\alpha_{\mu\nu} \partial_\alpha$
[ ] $\partial_\mu \partial_\nu = \Gamma^\alpha{}_{\mu\nu} \partial_\alpha$
[ ] $\partial_\mu \partial_\nu = \Gamma^\alpha_{\nu\mu} \partial_\alpha$
[ ] $\partial_\mu \partial_\nu = \Gamma^\alpha{}_{\nu\mu} \partial_\alpha$

(e) Which statements are false?

[ ] $\Gamma^\mu{}_{\alpha\beta}$ is a tensor.
[ ] $\Gamma^\mu{}_{\alpha\beta}$ is not a tensor.
[ ] The index α of $\Gamma^\mu{}_{\alpha\beta}$ is the one w.r.t. which we differentiate.
[ ] The index β of $\Gamma^\mu{}_{\alpha\beta}$ is the one w.r.t. which we differentiate.

2 The covariant derivative

The covariant derivative of a vector v^μ and a covector ω_μ is defined as

$$\nabla_\mu v^\nu \equiv \partial_\mu v^\nu + \Gamma^\nu{}_{\mu\alpha} v^\alpha\,, \qquad \nabla_\mu \lambda_\nu \equiv \partial_\mu \lambda_\nu - \Gamma^\alpha{}_{\mu\nu} \lambda_\alpha\,. \tag{1}$$

(a) Write down the explicit expression for $\nabla_\rho T^\mu{}_\nu$, where $T^\mu{}_\nu$ are the components of a $\binom{1}{1}$ tensor.

(b) Write down the explicit expression for $\nabla_\rho g_{\mu\nu}$, where $g_{\mu\nu}$ are the components of a $\binom{0}{2}$ tensor.

(c) For a vector v^μ, compute the second covariant derivative $\nabla_\alpha \nabla_\beta\, v^\mu$.

(d) For the same vector v^μ, compute $(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\, v^\mu$.

(e) Write the result of (d) in the form $(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\, v^\mu = R_{\alpha\beta}{}^\mu{}_\nu v^\nu - T_{\alpha\beta}{}^\lambda \nabla_\lambda v^\mu$ and read off the coefficients $R_{\alpha\beta}{}^\mu{}_\nu$ and $T_{\alpha\beta}{}^\lambda$. You should find the following result:

$$R_{\alpha\beta}{}^\mu{}_\nu = \partial_\alpha \Gamma^\mu{}_{\beta\nu} - \partial_\beta \Gamma^\mu{}_{\alpha\nu} + \Gamma^\mu{}_{\alpha\lambda} \Gamma^\lambda{}_{\beta\nu} - \Gamma^\mu{}_{\beta\lambda} \Gamma^\lambda{}_{\alpha\nu}\,, \tag{2}$$

$$T_{\alpha\beta}{}^\lambda = \Gamma^\lambda{}_{\alpha\beta} - \Gamma^\lambda{}_{\beta\alpha}\,. \tag{3}$$

The object $R_{\alpha\beta}{}^\mu{}_\nu$ is called the *Riemann curvature tensor*, and $T_{\alpha\beta}{}^\mu$ is the *torsion tensor*.