when	weekly, 2pm, 4-285 CCIS
contact	Jens Boos (boos@ualberta.ca)
URL	http://www.spintwo.net/Courses/Gauge-Theory-Student-Meetings/

4 Noether's theorem

Emmy Noether's famous theorem [1] lies at the foundations of gauge theory because it relates continuous global symmetries to conservation laws for matter currents. In these notes we explore the field-theoretical version of Noether's theorem.

4.1 Statement

Consider the action S for a collection of fields $\phi^A(x)$ (where A is an abstract index that can be used to treat multiple fields at once, and $\mathscr{M}$ is Minkowski space):

$$S = \int\limits_{\mathcal{M}} \mathrm{d}^4 x \, \mathscr{L}(\phi^A, \partial_\mu \phi^A, x^\mu) \,. \tag{1}$$

Suppose that this action is invariant under a global, continuous symmetry transformation

$$x^\mu \to x'^\mu = x^\mu + \delta x^\mu \,, \qquad \phi^A(x) \to \phi'^A(x') = \phi^A(x) + \delta\phi^A(x) \,, \tag{2}$$

where δx^μ is an arbitrary shift in the coordinates, and $\delta\phi^A$ is the total change in the field ϕ^A resulting from that shift in the coordinates as well as a dedicated transformation of the fields themselves,

$$\delta\phi^A = \bar{\delta}\phi^A + \boldsymbol{L}_{\delta x}\phi^A \,. \tag{3}$$

Let us call "$\delta\phi^A$" the *total field variation*, and let us call "$\bar{\delta}\phi^A$" the *same-point field variation*. It can be defined via

$$\phi'^A(x) = \phi^A(x) + \bar{\delta}\phi^A(x) \,, \tag{4}$$

where the main difference to Eq. (2) is that the field ϕ is evaluated at the old position x and not the shifted position x'. Also, in the above, $\boldsymbol{L}_v\phi^A$ denotes the Lie derivative of ϕ^A in the direction of the deformation δx^μ. It will become clear further below how this expression is evaluated. Noether's theorem then states that the following current is conserved on-shell:

$$j^\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^A)} \bar{\delta}\phi^A + \mathscr{L}\delta x^\mu \,. \tag{5}$$

By *on-shell* we mean that the equations of motions for ϕ^A are assumed to be satisfied, that is,

$$\frac{\partial \mathscr{L}}{\partial \phi^A} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^A)} = 0\,. \tag{6}$$

These equations of motion are also sometimes referred to as Euler–Lagrange equations for ϕ^A since they are derived from the variational principle as applied to the action S.

4.2 Derivation

Let us now prove Noether's theorem. First off, we know that the change in the action has to vanish by construction (since we are applying a symmetry transformation). Let us calculate this change in the action explicitly, for an infinitesimal transformation:

$$\begin{aligned}
0 = \delta S &= S[\phi'^A(x')] - S[\phi(x)] \qquad (7)\\
&= \int\limits_{\mathcal{M}'} \mathrm{d}^4x' \mathscr{L}\left(\phi'^A(x'), \partial_{\mu'}\phi'^A(x'), x'\right) - \int\limits_{\mathcal{M}} \mathrm{d}^4x \mathscr{L}\left(\phi^A(x), \partial_\mu \phi^A(x), x\right)\\
&= \int\limits_{\mathcal{M}} \mathrm{d}^4x \Big\{ \Big[\mathscr{L}\left(\phi'^A(x), \partial_\mu \phi'^A(x), x\right) - \mathscr{L}\left(\phi^A(x), \partial_\mu \phi^A(x), x\right) \Big] + \partial_\mu \Big[\mathscr{L}\left(\phi^A(x), \partial_\mu \phi^A(x), x\right) \delta x^\mu \Big] \Big\}\\
&= \int\limits_{\mathcal{M}} \mathrm{d}^4x \Big\{ \Big[\frac{\partial \mathscr{L}}{\partial \phi^A} \bar{\delta}\phi^A + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^A)} \bar{\delta}\partial_\mu \phi^A \Big] + \partial_\mu \Big[\mathscr{L} \delta x^\mu \Big] \Big\}\\
&= \int\limits_{\mathcal{M}} \mathrm{d}^4x \Big\{ \underbrace{\Big[\frac{\partial \mathscr{L}}{\partial \phi^A} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^A)} \Big]}_{\stackrel{\star}{=} 0 \text{ on-shell}} \bar{\delta}\phi^A + \partial_\mu \Big[\mathscr{L} \delta x^\mu + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^A)} \bar{\delta}\phi^A \Big] \Big\}\\
&\stackrel{\star}{=} \int\limits_{\mathcal{M}} \mathrm{d}^4x \partial_\mu \Big[\mathscr{L} \delta x^\mu + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^A)} \bar{\delta}\phi^A \Big]\,.
\end{aligned}$$

This deserves some explanation. From the second to the third line, we expressed the integral over $\mathscr{M}'$ in terms of an integral over $\mathscr{M}$ using the following identity:

$$\int\limits_{\mathcal{M}'} \mathrm{d}^4x' f(x') = \int\limits_{\mathcal{M}} \mathrm{d}^4x \left| \frac{\partial x'^\mu}{\partial x^\nu} \right| f(x) = \int\limits_{\mathcal{M}} \mathrm{d}^4x \big[f(x) + \partial_\mu \left(f(x) \delta x^\mu \right) \big]\,. \tag{8}$$

Then, in the third line all expressions only depend on x and not on x', so that we can pull everything under one integral. Leaving the total derivative in the last term alone, we from the third to the fourth line we expressed the difference of the two Lagrange densities in terms of the same-point field variations $\bar{\delta}\phi^A$. We also now omit all arguments of the Lagrange densities $\mathscr{L}$ since they now all depend on the original fields ϕ^A without any modifications. In the last step we integrate by parts and use the fact that the derivative ∂_μ and the same-point field variations commute,

$$\bar{\delta}\partial_\mu\phi^A = \partial_\mu\bar{\delta}\phi^A \,. \tag{9}$$

The above formula, where $\bar{\delta}$ is replaced by just δ is *not correct*. After performing the integration by parts we add the total derivative to the second term, and realize that the underbraced term vanishes on-shell, which concludes the derivation:

The entire expression has to vanish. Since the equal signs hold for any region $\mathscr{M}$ and thus cannot depend on the choice of it, we conclude that the divergence of the current j^μ has to vanish identically.

4.3 Some examples

This derivation was a bit formal, so let us get our hands dirty by calculating some conserved currents for some well-known examples. We will both treat symmetry transformations in the field ϕ^A as well as coordinate symmetries in x^μ.

4.3.1 Electric current of the complex scalar field

As we know quite well by now, the complex scalar field has the Lagrangian density

$$\mathscr{L} = -(\partial_\mu\phi)(\partial^\mu\phi^*) - V(|\phi|^2)\,, \tag{10}$$

where the potential term can be any function of $|\phi|^2$. This Lagrange density is invariant under the global continuous U(1) transformation

$$\phi'(x) = e^{i\alpha}\phi(x)\,, \quad \phi^{*\prime}(x) = e^{-i\alpha}\phi^*(x)\,, \quad \alpha \in \mathbb{R} = \text{const}\,. \tag{11}$$

We should remember now that this is a complex field, hence our abstract index A from above can be used to label ϕ and its complex conjugate ϕ^* such that

$$\phi^1 := \phi\,, \quad \phi^2 := \phi^*\,. \tag{12}$$

The infinitesimal version of the U(1) symmetry, expressed in the above language, takes the form

$$\bar{\delta}\phi^1 = \delta\phi^1 = i\alpha\phi\,, \quad \bar{\delta}\phi^2 = \delta\phi^2 = -i\alpha\phi^*\,, \quad \delta x^\mu = 0\,. \tag{13}$$

(Remember that the superscript "2" is not a power two, but rather the index $A = 2$.) We can now apply Noether's theorem and find for the conserved current

$$j^\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^A)}\bar{\delta}\phi^A = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^1)}\bar{\delta}\phi^1 + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^2)}\bar{\delta}\phi^2 = -\partial^\mu \phi^* i\alpha\phi - \partial^\mu\phi(-i)\alpha\phi^* = i\alpha\left(\phi^*\partial^\mu - \phi\partial^\mu\phi^*\right) . \tag{14}$$

Up to the leading constant factor of α, this precisely corresponds to the electric current we found in the previous meetings. Of course the physical current should not depend on the parameter arbitrary α, and hence we define the physical current to be (including a conventional minus sign)

$$j^\mu_{\text{phys.}} := -\left.\frac{\mathrm{d}j^\mu}{\mathrm{d}\alpha}\right|_{\alpha=0} = i\left(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi\right) . \tag{15}$$

The minus sign is just cosmetics, but we should remember that for each continuous parameter we obtain one conserved current.

4.3.2 Electric current of the Dirac fermion

The Dirac Lagrange density reads

$$\mathscr{L} = i\overline{\psi}\left(\gamma^\mu\partial_\mu + m\right)\psi\,. \tag{16}$$

Again, this Lagrange density is invariant under the global continuous U(1) transformation

$$\psi'(x) = e^{i\alpha}\psi(x)\,, \quad \overline{\psi}'(x) = e^{-i\alpha}\overline{\psi}(x)\,, \quad \alpha \in \mathbb{R} = \text{const}\,. \tag{17}$$

We should remember now that this is a complex field, hence our abstract index A from above can be used to label ψ and its adjoint spinor $\overline{\psi} := \psi^\dagger\gamma^0$ such that

$$\phi^1 := \psi\,, \quad \phi^2 := \overline{\psi}\,. \tag{18}$$

The infinitesimal version of the U(1) symmetry can be expressed as

$$\bar{\delta}\phi^1 = \delta\phi^1 = i\alpha\psi\,, \quad \bar{\delta}\phi^2 = \delta\phi^2 = -i\alpha\overline{\psi}\,, \quad \delta x^\mu = 0\,. \tag{19}$$

Then, the Noether current then is

$$j^\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^1)}\bar{\delta}\phi^1 + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^2)}\bar{\delta}\phi^2 = i\overline{\psi}\gamma^\mu i\alpha\psi + 0 = -\alpha\overline{\psi}\gamma^\mu\psi\,, \tag{20}$$

which again corresponds to the expressions we found earlier, up to the leading factor of $-\alpha$. The physical current, by definition, strips away this extra factor, and one obtains the well-known expression

$$j^\mu_{\text{phys.}} = \overline{\psi}\gamma^\mu\psi\,. \tag{21}$$

4.3.3 Energy-momentum of a real scalar field

Let us now forget about symmetry transformations on the fields themselves. What is left? Only empty spacetime itself. If we are talking about a relativistic field theory, it is usually formulated on Minkowski space $\mathcal{M}$: empty, three-dimensional space endowed with an additional time direction. This space is translationally invariant in any direction. This translation transformation corresponds to

$$\delta x^\mu = \epsilon^\mu\,, \tag{22}$$

where we should think of ϵ^μ as an infinitesimal, constant vector with four components. Let us consider, for simplicity, a real scalar field $\phi(x)$ with the Lagrange density

$$\mathscr{L} = -\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - V(\phi)\,, \tag{23}$$

where the potential depends on just ϕ and not x explicitly. Note that the global U(1) symmetry transformation (11) is *not* a symmetry of the scalar Lagrange density (23). Accordingly we will set

$$0 = \delta\phi = \bar{\delta}\phi + \boldsymbol{L}_{\delta x}\phi = \bar{\delta}\phi + \delta x^\mu\partial_\mu\phi = \bar{\delta}\phi + \epsilon^\mu\partial_\mu\phi \quad \Leftrightarrow \quad \bar{\delta}\phi = -\epsilon^\nu\partial_\nu\phi \neq 0\,, \tag{24}$$

where in the second step we inserted the definition of the Lie derivative, as demanded per Eq. (3). We can now calculate and find for the Noether current

$$j^\mu = \mathscr{L}\delta x^\mu + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\bar{\delta}\phi = \mathscr{L}\epsilon^\mu + (\partial^\mu\phi)\epsilon^\nu(\partial_\nu\phi) = \left[\mathscr{L}\delta^\mu_\nu + (\partial^\mu\phi)(\partial_\nu\phi)\right]\epsilon^\nu =: T^\mu{}_\nu\epsilon^\nu\,. \tag{25}$$

In the last equality we inserted a Kronecker delta δ^μ_ν in order to factor out the four continuous translation parameters ϵ^μ, and we see that in this case we do not obtain just one current, but in fact four currents, encoded in the energy-momentum tensor $T_{\mu\nu}$. The physical current (here we do not need the overall minus sign) is then given by

$$j^\mu{}_{\nu\text{phys.}} = +\left.\frac{\mathrm{d}j^\mu}{\mathrm{d}\epsilon^\mu}\right|_{\epsilon^\mu=0} = \mathscr{L}\delta^\mu_\nu + (\partial^\mu\phi)(\partial_\nu\phi) = T^\mu{}_\nu\,. \tag{26}$$

We see: translational invariance of Minkowski space gives rise to the conservation of energy and momentum. This translational invariance is also sometimes called an *isometry* of Minkowski space. Symmetries in the fields are also sometimes referred to as "internal symmetries" whereas symmetries of spacetime itself, like the translations in the present case, are then called "external symmetries."

4.3.4 Angular momentum of a real scalar field

To conclude this set of examples, let us now remember that Minkowski space is also invariant under Lorentz transformations, whose continuous parts in turn consist of three boosts as well as three spatial rotations. It is true that spatial reflections can also be considered as Lorentz transformations, but they are not continuous. And for Noether's theorem, as we derived at the beginning of this section, we can only employ continuous transformations.

An infinitesimal Lorentz transformation is described by the following coordinate transformation:

$$x'^{\mu} = x^{\mu} + \omega^{\mu}{}_{\nu} x^{\nu} \,, \quad \delta x^{\mu} = \omega^{\mu}{}_{\nu} x^{\nu} \,, \quad \omega_{\mu\nu} = -\omega_{\nu\mu} \in \mathbb{R} = \text{const.} \tag{27}$$

At first glance it looks like there is something fishy going on: the coordinate shift δx^{μ} depends on x^{μ}, so how can this be a global transformation? This is fine, as it turns out. The fact that we are talking about a global transformation is encoded in the fact that the antisymmetric matrix $\omega_{\mu\nu}$ is constant and does not depend on x^{μ}. Let us see how this works out.

Remember that during this coordinate transformation we do not change the field itself, $\delta\phi = 0$. As in the previous example this implies a non-zero same-point variation of the field,

$$0 = \delta\phi = \bar{\delta}\phi + \boldsymbol{L}_{\delta x}\phi = \bar{\delta}\phi + \delta x^{\mu}\partial_{\mu}\phi = \bar{\delta}\phi + \omega^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}\phi \quad \Leftrightarrow \quad \bar{\delta}\phi = -\omega^{\nu}{}_{\sigma}x^{\nu}\partial_{\sigma}\phi \neq 0 \,. \tag{28}$$

We now switch on the Noether machine and calculate

$$j^{\mu} = \mathscr{L}\delta x^{\mu} + \frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\phi)}\bar{\delta}\phi = \mathscr{L}\omega^{\mu}{}_{\nu}x^{\nu} + (\partial^{\mu}\phi)\omega^{\nu}{}_{\sigma}x^{\sigma}(\partial_{\nu}\phi) = \left[\mathscr{L}x_{\sigma}\delta^{\mu}_{\rho} + (\partial^{\mu}\phi)x_{\sigma}(\partial_{\rho}\phi)\right]\omega^{\rho\sigma} =: J^{\mu}{}_{\rho\sigma}\omega^{\rho\sigma} \,. \tag{29}$$

Again, we factored out the parameters $\omega^{\rho\sigma}$. We can now calculate the physical current by differentiating with respect to these parameters (again, we do not need the minus sign). However, we should remember that these parameters are combined into an antisymmetric matrix, $\omega^{\rho\sigma} = -\omega^{\sigma\rho}$. This means that after differentiating with respect to $\omega^{\rho\sigma}$ the result has to be antisymmetric under the exchange of the indices $\rho \leftrightarrow \sigma$. We find

$$j^{\mu}{}_{\rho\sigma\,\text{phys.}} = \left.\frac{\mathrm{d}j^{\mu}}{\mathrm{d}\omega^{\rho\sigma}}\right|_{\omega^{\rho\sigma}=0} = \frac{\mathrm{d}}{\mathrm{d}\omega^{\rho\sigma}}\left(\mathscr{L}x_{[\beta}\delta^{\mu}_{\alpha]} + (\partial^{\mu}\phi)x_{[\beta}\partial_{\alpha]}\phi\right)\omega^{\alpha\beta}\Big|_{\omega^{\rho\sigma}=0} = \mathscr{L}x_{[\sigma}\delta^{\mu}_{\rho]} + (\partial^{\mu}\phi)x_{[\sigma}\partial_{\rho]}\phi \tag{30}$$

$$= T^{\mu}{}_{[\rho}x_{\sigma]} = J^{\mu}{}_{\rho\sigma} \,. \tag{31}$$

In the above, we have used the "bracket notation" for antisymmetrization: $x_{[\mu}y_{\nu]} = \frac{1}{2}\left(x_{\mu}y_{\nu} - x_{\nu}y_{\mu}\right)$. The result is interesting: the physical current is a tensor of rank 3. This was somewhat expected:

- electric current j^{μ}: one current, one gauge parameter α
- energy-momentum current $T^{\mu}{}_{\nu}$: four currents, four gauge parameters ϵ^{μ}

- angular momentum current ${J^\mu}_{\rho\sigma}$: six currents, six gauge parameters $\omega_{\mu\nu} = -\omega_{\nu\mu}$

Also, and that is perhaps the most surprising result, the angular momentum of the scalar field ${J^\mu}_{\rho\sigma}$ can be expressed in terms of its own energy-momentum tensor ${T^\mu}_\nu$! What does that mean?

In these meetings we are talking about classical physics, but we should not forget that a real scalar field is the simplest possible model of matter, with no internal structure. In particular, the quantized version of it has spin zero. Since spin is related to angular momentum, we can think of the above result as follows: because the total angular momentum of the scalar field is just given by the antisymmetric product of ${T^\mu}_\nu$ and x_ρ (looks like $\vec{J} = \vec{r} \times \vec{p}$, doesn't it?) without any additional ingredients, the field ϕ has no internal angular momentum (i.e. no spin). Another way to put this: the angular momentum of the real scalar field ϕ is specified entirely by its orbital angular momentum, it has no intrinsic contributions.

In the case of fermions, which we may treat later in more detail, this will no longer be the case: its angular momentum is the sum of orbital angular momentum (similar to the scalar case) as well as an intrinsic contribution, which can be attributed to the spin-1/2 nature of fermions.

jb, meeting-4-v1.tex, Oct 4, 2018.

References

[1] E. Noether, "Invariante Variationsprobleme," Gött. Nachr. (1918) 235.