# How to derive these Lie Series formulas

**[+2] [2] Han de Bruijn**
**[2015-10-05 10:37:36]**
**[ lie-groups exponentiation differential-operators ]**
**[ https://math.stackexchange.com/questions/1465315/how-to-derive-these-lie-series-formulas ]**

Relates issues:

- How to properly apply the Lie Series [1]
- Exponential of a function times derivative [2]

In my old notes about Lie groups and/or operator calculus, I've encountered the following formulas:

$$e^{\lambda\, x^2\, \frac{d}{dx}}\, f(x) = f\left(\frac{x}{1-\lambda\, x}\right)$$
$$e^{\lambda\, \frac{1}{x}\, \frac{d}{dx}}\, f(x) = f\left(\sqrt{x^2+2\lambda}\right)$$
$$e^{\lambda\, x^3\, \frac{d}{dx}}\, f(x) = f\left(\frac{x}{\sqrt{1-2\lambda\, x^2}}\right)$$

I know how to derive the first one, but have no idea how I did the two others at that time.
Please help to refresh my memory.

[1] http://math.stackexchange.com/questions/1432104/how-to-properly-apply-the-lie-series
[2] http://math.stackexchange.com/questions/719487/exponential-of-a-function-times-derivative

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**[+4] [2015-10-05 16:02:11] Han de Bruijn**

The aim of this answer is to make the question self-contained, and propose a generalization of the answer by Sangchul Lee [1]. The basic formula to be employed is (anticipating with $y$ instead of $x$ and $g$ instead of $f$):

$$e^{\lambda \frac{d}{dy}}\, g(y) = g(y+\lambda)$$

Which is easily proved with Taylor series expansions for differential operators and functions:

$$e^{\lambda \frac{d}{dy}}\, g(y) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\lambda\, \frac{d}{dy}\right)^k g(y) = \sum_{k=0}^{\infty} \frac{g^{(k)}(y)}{k!}\lambda^k = g(y+\lambda)$$

**Generalization.** Substitute $y = \phi(x)$ , then:

$$\frac{d}{dy} = \frac{d}{d\phi(x)} = \frac{dx}{d\phi(x)}\frac{d}{dx} = \frac{1}{\phi'(x)}\frac{d}{dx}$$

Also notice that $g(y) = g(\phi(x)) = f(x) = f(\phi^{-1}(y))$. Thus:

$$e^{\lambda \frac{d}{dy}}\, g(y) = e^{\frac{\lambda}{\phi'(x)}\frac{d}{dx}} f(x) = g(y+\lambda) = f(\phi^{-1}(\phi(x)+\lambda))$$

Third formula in the question as a **specialization** of the above generalization:

$$1/\phi'(x) = x^3 \quad \Longrightarrow \quad \phi(x) = \int \frac{dx}{x^3} = \frac{-1}{2x^2} = y \quad \Longrightarrow \quad x = \sqrt{\frac{-1}{2y}} \quad \Longrightarrow$$

$$\phi^{-1}(x) = \sqrt{\frac{-1}{2x}} \quad \Longrightarrow \quad \phi^{-1}(\phi(x)+\lambda) = \sqrt{\frac{-1}{2(-1/(2x^2)+\lambda)}} = \frac{x}{\sqrt{1-2\lambda x^2}}$$

Hence it follows that:

$$e^{\lambda\, x^3 \frac{d}{dx}} f(x) = f\left(\frac{x}{\sqrt{1-2\lambda\, x^2}}\right)$$

But, with the general formula, another old puzzle is now solved completely ! Let $g$ be an arbitrary (real-valued, neat) function. Consider a Lie series as in Exponential of a function times derivative [2] :

$$e^{g(x)\partial} f(x) \qquad \text{with} \quad \partial = \frac{d}{dx}$$

First solve the differential equation:

$$g(x) = \frac{1}{\phi'(x)} \quad \Longrightarrow \quad \phi(x) = \int \frac{dx}{g(x)}$$

Then we have:

$$\boxed{e^{g(x)\partial} f(x) = f(\phi^{-1}(\phi(x)+1))}$$

[1] https://math.stackexchange.com/users/9340/sangchul-lee
[2] https://math.stackexchange.com/questions/719487/exponential-of-a-function-times-derivative

This sounds okay for me. :) - **Sangchul Lee**

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### [+2] [2015-10-05 11:23:22] Sangchul Lee [✔ACCEPTED]

Let $y = \frac{1}{2}x^2$. Then $\frac{1}{x}\frac{d}{dx} = \frac{d}{dy}$. Also notice that $f(x) = f(\sqrt{2y})$. Thus

$$e^{\frac{\lambda}{x}\frac{d}{dx}} f(x) = e^{\lambda\frac{d}{dy}} f(\sqrt{2y}) = f(\sqrt{2y+2\lambda}) = f(\sqrt{x^2+2\lambda}).$$

Second one can be solved in a similar way.

Please notice my generalization of your answer. Hope it is correct. Thanks. - **Han de Bruijn**
Maybe I'm picky, but if $y = \frac{1}{2}x^2$, then you have the two solutions $x = \pm\sqrt{2y}$. - **Wauzl**

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