# Exponential of a function times derivative

**[+12] [2] Danijel**
**[2014-03-20 07:43:34]**
**[ calculus derivatives operator-theory exponentiation ]**
**[ https://math.stackexchange.com/questions/719487/exponential-of-a-function-times-derivative ]**

Exponential of a derivative $e^{a\partial}$ is simply a shift operator, i.e.

$$e^{a\partial} f(x) = f(a+x)$$

This can be easily verified from a Taylor series

$$e^{a\partial} = \sum_{n=0}^{\infty} \frac{(a\partial)^n}{n!}$$

and

$$f(x) = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}$$

and applying one on another by using $\partial^m x^n = \frac{n!}{(n-m)!} x^{n-m}$ for $n \geq m$ and zero otherwise.

What if, instead of constant $a$, there is a function $g(x)$? In other words, I'm looking for $e^{g(x)\partial} f(x)$. Now, the derivative operator also acts on $g(x)$, which makes things very complicated and and seemingly intractable. For example, $[g(x)\partial]^2 = g(x)[g'(x) + g(x)\partial]\partial$ and it gets worse for higher orders.

Also, where can I find the list of identities such as $e^{a\partial} f(x) = f(a+x)$? Searching for function (or exponential or logarithm) of a derivative gets clogged with results about derivative of a function (or exponential or logarithm).

Have you checked out Fa di Bruno's formula ? - **Daniel Geisler**
@DanielGeisler There is no composition of function in this problem, so I doubt that Faà di Bruno's formula could help. - **Danijel**

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**[+7] [2014-04-01 19:26:41] Han de Bruijn [✔ACCEPTED]**

Instead of translation of a function $f(x) \to f(x+a)$ , let us consider *scaling*. This means that we are going to make intervals of the independent variable smaller, or larger, with a factor $\lambda > 0$. The transformed function is then defined by:

$$f_\lambda(x) = f(\lambda\, x)$$

Like with translations, it would be nice to develop the function $f_\lambda(x)$ into a Taylor series expansion around the original $f(x)$. But this is not as simple as in the former case. Unless some clever trick is devised, which reads as follows. Define a couple of new variables, $a$ and $y$, and a new function $\phi$ :

$$\lambda = e^a \qquad ; \qquad x = e^y \qquad ; \qquad \phi(y) = f(e^y)$$

Then, indeed, we can develop something into a Taylor series:

$$f_\lambda(x) = f(e^a\, e^y) = f(e^{a+y}) = \phi(y+a) = e^{a\frac{d}{dy}} \phi(y)$$

A variable such as $y$, which renders the transformation to be like a translation, is called a *canonical* variable. In the case of a scaling transformation, the canonical variable is obtained by taking the logarithm of the independent variable: $y = \ln(x)$. Working back to the original variables and the original function:

$$\phi(y) = f(e^y) = f(x) \qquad ; \qquad a = ln(\lambda)$$

$$\frac{d}{dy} = \frac{dx}{dy}\frac{d}{dx} = e^y \frac{d}{dx} = x\frac{d}{dx}$$

Where the operator $x\,d/dx$ is called the *infinitesimal operator* of a scaling transformation. Such an infinitesimal operator always equals differentiation to the canonical variable, which converts the transformation into a translation. We have already met, of course, the infinitesimal operator for the translations themselves, which is simply given by $(d/dx)$. This leads rather quickly to the following:

$$f_\lambda(x) = f(\lambda\, x) = e^{\ln(\lambda)\, x \frac{d}{dx}} f(x)$$

Which is somewhat bogus, because of some artificial restrictions imposed by our heuristics: $x = e^y$ had to be positive, for example. So let's specify this for the scaling transformation of $x$ itself, which is represented by the series $e^{\ln(\lambda)\, x\, d/dx} x$ :

$$e^{ln(\lambda)\, x \frac{d}{dx}} x = x + \ln(\lambda)\, x \frac{dx}{dx} + \frac{1}{2}\ln^2(\lambda)\, x \frac{d(x\, dx/dx)}{dx} + \cdots$$

$$= \left[1 + \ln(\lambda) + \frac{1}{2}\ln^2(\lambda) + \cdots\right] x = e^{\ln(\lambda)} x = \lambda\, x$$

Similarly (Update):

$$e^{ln(\lambda)\, x \frac{d}{dx}} x^n = x^n + \ln(\lambda)\, x \frac{dx^n}{dx} + \frac{1}{2}\ln^2(\lambda)\, x \frac{d(x\, dx^n/dx)}{dx} + \cdots$$

$$= x^n + \ln(\lambda)\, n\, x^n + \frac{1}{2}\ln^2(\lambda)\, n^2\, x^n + \frac{1}{6}\ln^3(\lambda)\, n^3\, x^n + \cdots$$

$$= \left[1 + \ln(\lambda^n) + \frac{1}{2}\ln^2(\lambda^n) + \frac{1}{6}\ln^3(\lambda^n) + \cdots\right] x^n = e^{\ln(\lambda^n)} x^n = \lambda^n\, x^n$$

Suppose that $f(x)$ can be written as a Taylor series expansion, then for all $x \in \mathbb{R}$ :

$$e^{ln(\lambda)\, x \frac{d}{dx}} \left[a_0 + a_1 x + a_2 \frac{1}{2} x^2 + \cdots\right] = a_0 + a_1(\lambda\, x) + a_2 \frac{1}{2}(\lambda\, x)^2 + \cdots$$

$$\Longrightarrow \qquad e^{ln(\lambda)\, x \frac{d}{dx}} f(x) = f(\lambda\, x)$$

(End of update) Since $\lambda$ must be positive, there exists no continuous transition towards problems where values are, at the same time, inverted or *mirrored*, like in:

$$f_\lambda(x) = f(-\lambda\, x)$$

For this to happen, the scaling transformation would to have to pass through a point where things are contracted to zero:

$$f_\lambda(x) = f(0\, x)$$

This already reveals a glimpse of the *topological issues* which may be associated with **Lie Groups** : remember that keyword. To be honest, I haven't seen any other generalization of your problem in 1-D, except the above scaling example.

**Update.** Well, not really. After some digging in my old notes, I've found a little bit more.
Consider the operation $e^\alpha\, x$ with $\alpha = g(x)\frac{d}{dx}$ . Then by definition:

$$e^{\alpha\, x} = 1 + \alpha\, x + \frac{1}{2}\alpha\,(\alpha\, x) + \frac{1}{3}\alpha\left(\frac{1}{2}\alpha\,(\alpha\, x)\right) + \cdots$$

This can be written recursively as:

$$e^{\alpha}\, x = x + \alpha_1 x + \alpha_2 x + \alpha_3 x + \cdots \quad ; \quad \alpha_1 = \alpha \quad ; \quad \alpha_n = \frac{1}{n}\alpha\, \alpha_{n-1}$$

We have seen cases where $g(x) = a$ and $g(x) = \ln(\lambda)\, x$ . Now let's try another example, with $g(x) = x^2$ :

$$\begin{aligned} \alpha_1 x &= x^2 \frac{d}{dx} x = x^2 \\ \alpha_2 x &= \frac{1}{2} x^2 \frac{d}{dx} x^2 = x^3 \\ \alpha_3 x &= \frac{1}{3} x^2 \frac{d}{dx} x^3 = x^4 \\ &\cdots \\ \alpha_n &= \frac{1}{n} x^2 \frac{d}{dx} x^n = x^{n+1} \end{aligned}$$

Consequently, say for real $0 < x < 1$ :

$$e^{x^2\, d/dx} x = x + x^2 + x^3 + \cdots + x^n + \cdots = \frac{x}{1-x}$$

Which can perhaps be generalized to functions $f(x)$ that have a Taylor expansion.
One might think now that the above results may be combined as follows:

$$e^{(ax^2+bx+c)d/dx} x = e^{c\, d/dx} e^{bx\, d/dx} e^{ax^2\, d/dx} x = e^{ax^2\, d/dx} e^{bx\, d/dx} e^{c\, d/dx} x$$

But it can readily be verified that such is not the case. The reason is that the operators $x^2\, d/dx$ , $x\, d/dx$ , $d/dx$ do **not commute**. Define the *commutator* $[\, ,]$ of two operators $\alpha$ and $\beta$ as:

$$[\alpha\, , \beta] = \alpha\beta - \beta\alpha$$

Then prove that:

$$\left[x^2 \frac{d}{dx}, x\frac{d}{dx}\right] \neq 0 \quad ; \quad \left[x^2 \frac{d}{dx}, \frac{d}{dx}\right] \neq 0 \quad ; \quad \left[x \frac{d}{dx}, \frac{d}{dx}\right] \neq 0$$

**Late revision.** I've ordered the following book and reading it now:

- Sophus Lie, *Vorlesungen über Differentialgleichungen mit bekannten Infinitesimalen Transformationen*, bearbeitet und herausgegeben von Dr. Georg Wilhelm Scheffers,
  Leipzig (1891). Availability: Amazon [1] , bol.com [2] .

Formulated in somewhat outdated notation I find the following IMHO astonishing **Theorem** on page 50 and next. Operator notation is mine:

$$\boxed{e^{t\phi(x)\frac{d}{dx}} f(x) = f\left(e^{t\phi(x)\frac{d}{dx}} x\right)}$$

Here $\phi(x)$ and $f(x)$ are "neat" but for the rest quite arbitrary functions. Therefore the differential operator and the function are always *commutative*, which is quite a non-trivial fact. When applied to the last of the above examples (slightly modified) we have via page 75 of the book:

$$e^{tx^2 \frac{d}{dx}} f(x) = f\left(\frac{x}{1-xt}\right)$$

So it is indeed sufficient to apply the operator $\exp(t\phi(x)d/dx)$ to the independent variable $x$ only. If that results in a closed form, then you can apply the Theorem and have a closed form for any other function $f(x)$ as well.

**Sad remark.** The book by Georg Scheffers is abundant with "non rigorous" notions, especially *infinitesimals*. The latter are quite essential for understanding the book. For me, as a physicist by education, this represents no problem at all. But I know from bad experience that those good *old infinitesimals* represent sort of a taboo for modern mathematics. Therefore, in retrospect, it can be understood very well why this approach by Georg Scheffers hasn't found wide audience among professional mathematicians. Even worse. I find that professional mathematicians rather have distorted the original theory as meant by Sophus Lie a great deal. Such that essential parts of it, like the above Theorem, tend to be *erased from common mathematical knowledge*. Which I hope not.

[1] http://rads.stackoverflow.com/amzn/click/B009T8GNPA
[2] http://www.studieboekencenter.nl/aanbod/isbn-9785879251753-vorlesungen-uber-differentialgleichungen-mit-bekannter-infinitesimalen-transformationen-bearb-und-hrsg-von-georg-scheffers-german-edition

So, there's no general expression for $e^{g(x)\partial}f(x)$? Well, knowing the result $e^{ax\partial}f(x) = f(e^a x)$ is better than nothing, I guess. But, are you sure that something like $e^{i\pi x\partial}f(x) = f(-x)$ wouldn't work? - **Danijel**
Digging in my old notes about *Lie Groups* , it looks like there does exist some general "expression" for it. But that's only an impression. My notes are too unsystematic and my knowledge has become so rusty that being explicit about it almost certainly wouldn't help. - **Han de Bruijn**
(1) About the last part of your comment. Because of $i\pi = \ln(-1)$ I see no reason why $e^{i\pi x\partial}f(x) = f(-x)$ wouldn't work. Further calculation confirms indeed that $e^{i\pi x\partial}x$ equals $e^{i\pi}x = -x$ and so forth. Thanks for pointing this out. - **Han de Bruijn**
I noticed the terminology "canonical variable". Is there any connection with the canonical variables of Hamiltonian formalism of classical mechanics? (BTW, this answer deserves a lot more upvotes than just two.) - **Giuseppe Negro**
(1) @Danijel: There is "late revision" of the answer (sorry, I had to buy and read a book first) that might be of interest to you. - **Han de Bruijn**
@GiuseppeNegro: I don't think there is an intimate relationship, but my knowledge of the Hamiltonian formalism is quite rusty. A canonical variable in Lie Group Theory is the one that makes a (say one-parameter) group equivalent with a (1-D) translation. With e.g. planar rotations this means the introduction of polar coordinates (i.e. angular translation). - **Han de Bruijn**
@HandeBruijn, thanks for additional information! Is there also an English version of this book by Sophus Lie available? - **Danijel**
@Danijel: bol.com/nl/p/... ? - **Han de Bruijn**
Does there exist a similar formula for $e^{t\phi(x)\frac{d^2}{dx^2}}f(x)$, in particular for $e^{tx^2\frac{d^2}{dx^2}}f(x)$? - **user85503**

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**[+1] [2015-10-05 16:17:17] Han de Bruijn**

The question is answered affirmative (and in a much simpler way) elsewhere:

- How to derive these Lie Series formulas [1]

**Summary.** First solve the differential equation:

$$g(x) = \frac{1}{\phi'(x)} \quad \Longrightarrow \quad \phi(x) = \int \frac{dx}{g(x)}$$

Then we have (barring division by zero and other issues):

$$e^{g(x)\partial}f(x) = f(\phi^{-1}(\phi(x)+1))$$

**Update**, triggered by another question (but where?)
By definition, for a function $\phi$ and its inverse:

$$y = \phi(x) \quad \Longleftrightarrow \quad x = \phi^{-1}(y) \quad \Longleftrightarrow \quad \phi^{-1}(\phi(x)) = x$$

From this, an elementary result in calculus follows:

$$\frac{dy}{dx}\frac{dx}{dy} = 1 = \frac{d\phi(x)}{dx}\frac{d\phi^{-1}(y)}{dy} \quad \Longrightarrow$$
$$\frac{d\phi^{-1}(y)}{dy} = \frac{1}{\phi'(x)} = \frac{1}{\phi'(\phi^{-1}(y))} \quad \Longrightarrow$$
$$\frac{d\phi^{-1}(x)}{dx} = \frac{1}{\phi'(\phi^{-1}(x))}$$

There is an application with the Lie series. We have:

$$u(t) = e^{t\,g(x)\frac{d}{dx}}x = \phi^{-1}(\phi(x) + t) \quad \text{with} \quad g(x) = \frac{1}{\phi'(x)}$$

$$u(0) = e^{0\,g(x)\frac{d}{dx}}x = x = \phi^{-1}(\phi(x))$$

It follows that:

$$\frac{du}{dt} = \frac{d\phi^{-1}(\phi(x) + t)}{dt} = \frac{1}{\phi'(\phi^{-1}(\phi(x) + t))} = \frac{1}{\phi'(u(t))}$$

In short:

$$u(t) = e^{t\,g(x)\frac{d}{dx}}x \quad \Longleftrightarrow \quad \dot{u}(t) = g(u(t)) \quad \text{with} \quad x = u(0)$$

[1] https://math.stackexchange.com/questions/1465315/how-to-derive-these-lie-series-formulas

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