Searching for new physics with gravitational waves: let's build an extreme-mass ratio inspiral waveform generator!

KSETA Topical Course, Oct 7-10, 2024

Jens Boos[1],
Institute for Theoretical Physics, Karlsruhe Institute of Technology, D-76128 Karlsruhe, Germany

Abstract

Gravitational waves are the emergent medium to probe our understanding of gravity, with increasing relevance in the age of gravitational wave astronomy. Current experiments (LIGO/Virgo/Kagra) are sensitive to gravitational waves stemming from the collision of stellar-mass black holes of roughly comparable masses. While an excellent probe for gravity in the strong-field regime, analytical treatments are extremely difficult to perform and in practice one resorts to numerical relativity. However, future experiments, like the Laser Interferometer Space Antenna (LISA), will be sensitive to the gravitational waves stemming from the collision of small, stellar mass black holes with giant, supermassive black holes. In such "extreme-mass ratio inspirals" (for short: EMRIs) it is possible to perform perturbative computations, since the large mass ratio allows one to approximate the small orbiting black hole as a point particle. The goal of this course is to understand how a fundamental model of gravity (say, general relativity) can be used to estimate the shape of the gravitational waves stemming from such an EMRI, using perturbative and largely analytical techniques. To that end, we will develop a simple MATHEMATICA sheet that will generate such a gravitational wave pattern (to zeroth order, with necessary simplifications in resolution and accuracy). In a second step, we will consider a modified version of gravity, and explore how this qualitatively changes the gravitational wave pattern.

emri-101-lecture-notes-v3.tex, Oct 9, 2024, jb

Note: Graduate-level general relativity is helpful, but not required.

[1]Email: jens.boos@kit.edu

Contents

List of Figures

Chapter 0

Overview and motivation

We live in a fascinating time: gravitational waves, measured by terrestrial experiments, have indirectly confirmed the existence of astrophysical black holes. Moreover, interferometric imaging techniques have managed to capture snapshots of supermassive black holes. Due to their strong gravitational fields, black holes present a unique opportunity to put general relativity to the test. Is it really the "only game in town" or can there be subtle deviations?

The development of strong gravity tests is often performed with tools of numerical relativity, capable of describing the dynamics of neutron stars or the time-resolved merger of black holes. Analytical methods have not (yet?) been that successful. Unfortunately, gravitational waves in the 100Hz-range, as they are detectable by terrestrial experiments, are generated primarily by merging stellar-size black holes—the regime of numerical relativity. Lower-frequency gravitational waves in the millihertz domain, will be detectable by the Laser Interferometer Space Antenna (LISA) experiment. Fortunately for us, a predominant source for such gravitational radiation stems from the slow, often week-long inspiral of small, stellar-mass black holes, into a central, supermassive primary black hole. The upshot: the large separation in masses (often, the smaller black hole is 100-10,000 times less massive) allows for a perturbative description of that process. The goal of this lecture is simple: let us try to understand how we can perturbatively describe such an extreme mass ratio inspiral (EMRI) event, and how we can extract the gravitational wave signatures.

This research area has been developed for a long time, and we will not be able to perform cutting edge calculations to the best known precision. However, with relative ease, we will be able to extract leading order effects for non-rotating black holes. (Realistically, astrophysical black holes are always rotating; and while this does not immediately present an analytic intractability, it would significantly increase the complexity of equations.) For this reason we will focus on an approach described neatly in Babak *et al.* [1], adapted to the non-rotating case. I encourage you to have a look at this article, since it contains an excellent introduction into the topic and a very clear description of all steps involved. If you are interested in details of hands-on contemporary calculations, I refer you to Alexandra Hanselman's S.B. thesis [2] that covers many technical aspects from a self-contained, introductory perspective.

0.1 Main idea: how to generate EMRI waveforms

Before we delve into all the details (and may get lost in some formulae), I wanted to briefly describe the main steps necessary to generate gravitational wave waveforms for EMRIs. Here is a step-by-step list:

(i) Obtain the bound orbits.
We consider a test particle in a bound orbit around a black hole, and we find the equations that govern its motion. Since we are interested in observation of that motion, we rephrase everything in terms of the time coordinate t of a distant observer.

(ii) Calculate the gravitational wave radiation.
Using an approximation, we estimate the energy and angular momentum radiated away via gravitational waves. We will utilize linearized gravity for this, and for slow-moving sources it is moreover sufficient to utilize the quadrupole formula instead of the Press formula.

(iii) Obtain inspiral orbit.
In an adiabatic approximation, we elevate the orbiting particle's energy and angular momentum to time-dependent functions, taking into consideration the loss of energy and angular momentum due to gravitational waves. This decrease leads to a decaying orbit.

(iv) Compute gravitational wave signal.
With the inspiral orbit available, we consider a motion of a particle in flat spacetime following this curve, and extract the gravitational wave signal from this, again utilizing linearized gravity.

(v) Rinse an repeat.
We can now repeat this for different initial particle parameters, and generalize these methods to other geometries. Moreover, we can refine the analysis by taking into account subleading effects, such as black hole rotation, spin-curvature coupling, self forces, and more.

And that's it. Conceptually, it is surprisingly simple, but keep in mind that this is largely due to the fact that the large mass ratio allows us to make these simplifications. Eventually, when the orbiting particle gets too close to the primary black hole, our approximations break down and we would have to resort to numerical relativity to properly capture the black hole merger.

0.2 Organization of this lecture

Since not everybody has the same background in general relativity, I decided to add Chapter 1 that summarizes the main results relevant for this lecture. Needless to say, it is far from a self-contained introduction into general relativity (and slightly beyond), but I hope it can serve as a good first point of reference.

After the introduction, in Chapter 2 we will start talking about the spherically symmetric black hole solution of general relativity, the so-called Schwarzschild solution. We will discuss its symmetries with a particular focus on how we can use them to construct conserved quantities for particle motion, since that will greatly simplify the discussion of particle orbits. We will use some MATHEMATICA notebooks (available on the course website) to help us along with computations and some simple numerics.

With stable particle orbits under control, in Chapter 3 we will next briefly revisit the notion of linearized gravity, wherein it is rather straightforward to describe gravitational waves emanating from isolated gravitational systems (such as the black hole-black hole binary system formed from a supermassive black hole and a stellar mass black hole that we will discuss later). Here we will again be brief, but I will make sure that the main results (linearized field equation and quadrupole formula) are derived rigorously enough that any interested students can fill in the gaps at home. (If not, please contact me via email or send me an anonymous question (password: EMRI), and I will be happy to help as best as I can.

In the next step, in Chapter 4, we will combine the results: using the quadrupole formula, we will estimate how much energy and angular momentum is radiated away from the system, and compute its leading order effect on the particle orbit. Under influence of this backreaction, the orbit will then become an inspiral orbit. Taking that as an input, using the formulas from the previous chapter we will finally see an EMRI waveform. Again, tailored MATHEMATICA notebooks will help us with that.

Last, one of the goals of this lecture is to motivate how we can use EMRI waveforms to search for new physics. To see how this is, in principle, possible, we will consider a spherically symmetric black hole that carries a small dark charge, formally equivalent to the Reissner–Nordström black hole. After changing the metric wherever necessary, we will track how this extra term changes both the inspiral as well as the EMRI waveform.

Ultimately, I hope that this lecture will be interesting not just for gravitational physicists, but also get the general idea across to a broader audience: gravitational waves have come to stay, and it is time we learn how to use them to help us making discoveries in fundamental physics.

Chapter 1

Basics of general relativity and beyond

These lecture notes are by no means supposed to replace a proper introduction into the basic aspects of general relativity, but in this chapter we will review a few basic concepts that are most helpful for the topic of extreme mass ratio inspirals and their gravitational wave signature. Those topics include the metric, the notion of symmetries of metrics (called *isometries*), taking derivatives in a coordinate-invariant manner (via the *covariant* derivative), the motion of test particles in spacetime geometries on *geodesics*, as well as conserved quantities for such particle motion in the presence of symmetries, which corresponds to a manifestation of Noether's theorem in the context of gravity.

Our conventions are $(-, +, +, +)$ metric signature, and we will use units where the speed of light is set to unity, $c = 1$. This means that GM, where M is a mass, has dimensions of length.

1.1 The metric

Before we talk about the metric, let us fix some coordinates. We denote them by x^μ. In the context of these lecture notes, we will almost exclusively work in spherical coordinates, $x^\mu = \{t, r, \theta, \varphi\}$, but the whole point of general relativity is that it does not matter what coordinates you use—the final, physical results will not depend on that choice. When we want to evaluate tensor components (more on that below) we set indices to coordinate names, for example $\mu = t$ or $\mu = r$. (In some literature, we instead find $\mu = 0$ or $\mu = 1$, that is, numerical notation, and then we have to remember that $x^0 = t$ and $x^1 = r$ and so on. By using the coordinate names for the components we can avoid that sort of confusion.)

In general, a *tensor* is an object with p upper indices, and q lower indices that transforms homogeneously under coordinate transformations. We say that such a tensor $T^{\mu_1 \cdots \mu_p}{}_{\nu_1 \dots \nu_q}$ is of rank (p, q).

The *spacetime metric*, an important tensor for us in this lecture, is a rank-(0,2) tensor (meaning that it has two lower indices) that we will denote by $g_{\mu\nu}$, and it is the fundamental variable in general relativity. The *inverse metric* is denoted by $g^{\mu\nu}$ and it is related to the metric via

$$g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu_\nu\,, \tag{1}$$

where δ^μ_ν is the Kronecker symbol (equal to 1 if $\mu = \nu$ and 0 otherwise). In practice, we often denote the metric indirectly via the so-called *line element*,

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu\,. \tag{2}$$

This has the advantage that we can read off both the metric components and the coordinates being used. For example, the line element of flat spacetime expressed in spherical coordinates is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2\,, \tag{3}$$

and we can read off

$$g_{tt} = -1\,, \quad g_{rr} = 1\,, \quad g_{\theta\theta} = r^2\,, \quad g_{\varphi\varphi} = r^2\sin^2\theta\,. \tag{4}$$

The components of the inverse metric are then obtained simply by matrix inversion, which is particularly simple for this diagonal metric:

$$g^{tt} = -1\,, \quad g^{rr} = 1\,, \quad g^{\theta\theta} = \frac{1}{r^2}\,, \quad g^{\varphi\varphi} = \frac{1}{r^2\sin^2\theta}\,. \tag{5}$$

Sometimes we also find the following notation for the inverse line element in the literature:

$$\partial_s^2 = g^{\mu\nu}\partial_\mu\partial_\nu\,. \tag{6}$$

where the ∂_μ is a shorthand notation for $\partial/\partial x^\mu$. In the example case from above, we have the inverse line element

$$\partial_s^2 = -\partial_t^2 + \partial_r^2 + \frac{1}{r^2}\partial_\theta^2 + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2\,. \tag{7}$$

Why did we bother with all of this? One really important task is to successfully switch between different coordinate systems. Given an old coordinate system x^μ and a new coordinate system y^ν, we can compute the Jacobian and the inverse Jacobian,

$$\frac{\partial x^\mu}{\partial y^\nu}\,, \qquad \frac{\partial y^\mu}{\partial x^\nu}\,. \tag{8}$$

And a rank-(p,q) tensor transforms "homogeneously," by which we mean

$$T^{\mu_1\ldots\mu_p}{}_{\nu_1\ldots\nu_q} \to \frac{\partial y^{\mu_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial y^{\mu_p}}{\partial x^{\alpha_p}}\frac{\partial x^{\beta_1}}{\partial y^{\nu_1}}\cdots\frac{\partial x^{\beta_q}}{\partial y^{\nu_q}}T^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_q}\,. \tag{9}$$

That's a lot of indices! Thankfully, we almost never have to do these types of manipulations. For example, in the case of the metric, while you are of course free to use the above formula for transformations, it is easier to just use the expression for $\mathrm{d}s^2$ and apply the chain rule, and then read off the new components. For example, let us switch from spherical coordinates to cylindrical coordinates $y^\mu = \{t, z, \rho, \varphi\}$, with the identifications $\rho = r\sin\theta$ and $z = r\cos\theta$, we find

$$\mathrm{d}\rho = \sin\theta\mathrm{d}r + r\cos\theta\mathrm{d}\theta\,,\quad \mathrm{d}z = \cos\theta\mathrm{d}r - r\sin\theta\mathrm{d}\theta\,, \tag{10}$$

or, equivalently,

$$\mathrm{d}r = \cos\theta\mathrm{d}z + \sin\theta\mathrm{d}\rho\,,\quad \mathrm{d}\theta = \frac{1}{r}\left(\cos\theta\mathrm{d}\rho - \sin\theta\mathrm{d}z\right)\,. \tag{11}$$

We can now insert this directly into our metric,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2\,, \tag{12}$$

and after utilizing $\sin^2\theta + \cos^2\theta = 1$ we arrive at

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}\rho^2 + \rho^2\mathrm{d}\varphi^2\,. \tag{13}$$

Immediately we can read off, for example, that $g_{\rho\rho} = 1$ and that now $g_{\varphi\varphi} = \rho^2$. I find this method of coordinate-transforming a metric much easier than utilizing the more cumbersome (but equivalent) Jacobian method of Eq. (9).

Another important object are vector fields. We write them in a coordinate-invariant way as

$$V = V^\mu\partial_\mu\,. \tag{14}$$

Here, ∂_μ is the coordinate basis, and it makes an appearance similar to the relation for the inverse metric $g^{\mu\nu}$ in Eq. (7). This notation is a bit confusing if you are seeing it for the first time, but it is really nice since it allows us to transform between coordinates very easily. Say we have the vector field $V = \partial_r$ (that is, a unit vector in the radial direction) and we again transform to cylindrical coordinates. Here is how we do it (hint, hint: chain rule):

$$\begin{aligned} V = \frac{\partial}{\partial r} &= \frac{\partial z}{\partial r}\frac{\partial}{\partial z} + \frac{\partial \rho}{\partial r}\frac{\partial}{\partial \rho} = \cos\theta\,\partial_z + \sin\theta\,\partial_\rho \\ &= \frac{z}{r}\,\partial_z + \frac{\rho}{r}\,\partial_\rho = \frac{z}{\sqrt{z^2+\rho^2}}\,\partial_z + \frac{\rho}{\sqrt{z^2+\rho^2}}\,\partial_\rho\,. \end{aligned} \tag{15}$$

We see these kind of transformations all the time, and I encourage you to do an example yourself. If you take the vector $U = \partial_\theta$, for example, how does it look like in cylindrical or Cartesian coordinates? [**Do it!** ☺]

1.2 Isometries

The metric is the main protagonist in general relativity. And, already at this point, we can introduce concepts that will become very useful: Killing vectors, Lie derivatives, and isometries. Note that the covariant derivative, which we will talk about in the next section, is *not* needed for any of this discussion. Loosely speaking, an isometry of the metric means that there exists a coordinate direction in which the metric does not change. Let us parametrize this coordinate direction by a vector field ξ^μ. Then, the precise statement of an isometry of the metric in the direction of that vector field is

$$\mathcal{L}_\xi g_{\mu\nu} = 0\,. \tag{16}$$

We say: the Lie derivative of the metric vanishes along the vector field ξ. Here, the Lie derivative $\mathcal{L}$ is a type of derivative that can be defined for any tensors—all you need is a vector field ξ^μ along which you differentiate. For a (1,1)-tensor $T^\mu{}_\nu$ the Lie derivative is defined as

$$\mathcal{L}_\xi T^\mu{}_\nu = \xi^\alpha(\partial_\alpha T^\mu{}_\nu) - (\partial_\alpha \xi^\mu)T^\alpha{}_\nu + (\partial_\nu \xi^\alpha)T^\mu{}_\alpha\,. \tag{17}$$

Importantly, the Lie derivative of a tensor is also a tensor. The first term is always there; the second term, with a minus sign, takes care of the upper index; and the third term, with the plus sign, takes care of the lower index. If you want to differentiate an object with more than one upper index or lower index, then more of these terms have to be added (one for each index). For example, for the metric, which is a (0,2)-tensor, we write

$$\mathcal{L}_\xi g_{\mu\nu} = \xi^\alpha(\partial_\alpha g_{\mu\nu}) + (\partial_\mu \xi^\alpha)g_{\alpha\nu} + (\partial_\nu \xi^\alpha)g_{\mu\alpha}\,. \tag{18}$$

To understand why a vanishing Lie derivative can be interpreted as an isometry, imagine for now that the metric does not depend on a certain coordinate z. Then we can define the vector field

$$\xi^\mu = \delta^\mu_z \,, \tag{19}$$

which is zero in all directions except for in the z-direction. Inserting this into Eq. (18) and setting it to zero we find the relation

$$\mathcal{L}_\xi g_{\mu\nu} = \frac{\partial}{\partial z} g_{\mu\nu} = 0 \,, \tag{20}$$

which literally means that the metric does not depend on the coordinate z. The interpretation is not always as simple (as we will see when we talk about the rotational invariance of a spherically symmetric metric futher below in Sec. 2.1), but it is a good example to keep in mind when we get lost in mathematical details.

One last thing: The Lie derivative of a vector A^μ along a vector field B^μ has a special interpretation. Let us calculate it and see:

$$\mathcal{L}_B A^\mu = B^\alpha(\partial_\alpha A^\mu) - (\partial_\alpha B^\mu) A^\alpha \tag{21}$$

We can rearrange the terms a bit to emphasize a structure:

$$\mathcal{L}_B A^\mu = (B^\alpha \partial_\alpha) A^\mu - (A^\alpha \partial_\alpha) B^\mu \,. \tag{22}$$

Do you see it? Recalling Eq. (14), it looks like the difference between "vector B acting on A" and "vector A acting on B," doesn't it? Therefore it is antisymmetric under the exchange of A and B. This structure is so common that we give it a special symbol and name. It is the *commutator* of two vector fields:

$$[A, B]^\mu \equiv \mathcal{L}_A B^\mu = (A^\alpha \partial_\alpha) B^\mu - (B^\alpha \partial_\alpha) A^\mu \tag{23}$$

Why is this equation at the end of the section of isometries? If you have more than one Killing vector, it is often helpful to calculate the commutator between Killing vectors, to determine their algebra. This helps to understand the underlying symmetry group. Sure, sometimes it is obvious if you are in the right coordinate system. But remember that coordinates do not mean anything, and if you are in a wrong coordinate system, something that is entirely obvious in one coordinate system may not be obvious at all in the one you are currently using. Commutation relations, however, are coordinate independent, and hence they are your friend whenever you have a bunch of Killing vectors you want to understand better. And to drive home that point, we will do that very thing when we talk about spherical symmetry in Sec. 2.1.

1.3 The connection and the covariant derivative

Lie derivatives are nice, but loosely speaking we can only differentiate along vector fields. What if we want to have a general type of derivative? This is where the covariant derivative comes in.

We use the symbol ∇_μ for it, in contrast to the partial derivative ∂_μ. The usual problems with derivatives is that they do not commute with coordinate transformations: they destroy the homogeneous transformation law (9). For example, while $T^\nu{}_\rho$ is a rank-(1,1) tensor, $\partial_\mu T^\nu{}_\rho$ is *not* a rank-(1,2) tensor. However, the quantity $\nabla_\mu T^\nu{}_\rho$ *is* a rank-(1,2) tensor, and it is defined as

$$\nabla_\mu T^\nu{}_\rho \equiv \partial_\mu T^\nu{}_\rho + \Gamma^\nu{}_{\alpha\mu} T^\alpha{}_\rho - \Gamma^\alpha{}_{\rho\mu} T^\nu{}_\alpha \,. \tag{24}$$

For tensors of different rank, additional terms need to be added. For each upper index, a positive term is added (like the second one on the right-hand side), and for each lower index, a negative term is added (like the third term on the right-hand side). The new ingredient in the covariant derivative is the *connection* $\Gamma^\lambda{}_{\mu\nu}$, which is *not a tensor*. In order for the above to transform as a rank-(1,2) tensor, the connection itself has to transform as

$$\Gamma^\lambda{}_{\mu\nu} \to \frac{\partial y^\lambda}{\partial x^\alpha}\frac{\partial x^\beta}{\partial y^\mu}\frac{\partial x^\gamma}{\partial y^\nu}\Gamma^\alpha{}_{\beta\gamma} + \frac{\partial x^\lambda}{\partial y^\alpha}\frac{\partial^2 x^\alpha}{\partial y^\mu \partial y^\nu} \,. \tag{25}$$

Note that this transformation law is inhomogeneous! If you want to learn more about this concept, try to derive the above formula. What is remarkable is that we have not even talked about what the connection is—all that is required is the inhomogeneous transformation law, and that's it.

Part of the reason is that a connection is not necessarily unique. We need to impose conditions on it (or on the covariant derivative) to learn more about it, and to constrain its form a bit more. Since we are working in a setting with a spacetime metric, a somewhat natural choice is to demand that the covariant derivative of the metric vanishes,

$$0 = \nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^\alpha{}_{\nu\mu} g_{\alpha\rho} - \Gamma^\alpha{}_{\rho\mu} g_{\nu\alpha} \,. \tag{26}$$

Taking the combination $\nabla_\mu g_{\nu\rho} + \nabla_\rho g_{\mu\nu} - \nabla_\nu g_{\rho\mu} = 0$ and defining the objects

$$\widetilde{\Gamma}^\lambda{}_{\mu\nu} \equiv \frac{1}{2} g^{\lambda\alpha} \left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu} \right) \,, \tag{27}$$

$$T_{\mu\nu}{}^\lambda \equiv \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} \,, \tag{28}$$

$$K^\lambda{}_{\mu\nu} = \frac{1}{2} \left(T_\mu{}^\lambda{}_\nu + T_\nu{}^\lambda{}_\mu + T_{\mu\nu}{}^\lambda \right) \,, \tag{29}$$

the connection takes the simple form

$$\Gamma^\lambda{}_{\mu\nu} = \widetilde{\Gamma}^\lambda{}_{\mu\nu} + K^\lambda{}_{\mu\nu}\,. \tag{30}$$

Here, $\widetilde{\Gamma}^\lambda{}_{\mu\nu}$ is called the the Levi-Civita connection (or, less formally and for historical reasons, Christoffel symbols), and $K^\lambda{}_{\mu\nu}$ is called the contortion tensor, and $T_{\mu\nu}{}^\lambda$ is the torsion tensor. In general relativity, this tensor is assumed to be identically zero, and there the connection is identical to the Levi-Civita connection that is derived only from the metric.

Hence, if the connection is symmetric, the torsion tensor is zero. Note, however, that the converse is not true! Torsion enters the symmetric part of the connection as follows:

$$\Gamma^\lambda{}_{(\mu\nu)} \equiv \frac{1}{2}\left(\Gamma^\lambda{}_{\mu\nu} + \Gamma^\lambda{}_{\nu\mu}\right) = \widetilde{\Gamma}^\lambda{}_{\mu\nu} + \frac{1}{2}\left(T_\mu{}^\lambda{}_\nu + T_\nu{}^\lambda{}_\mu\right)\,. \tag{31}$$

While the contortion tensor $K^\lambda{}_{\mu\nu}$ is truly a tensor, the Levi-Civita connection $\widetilde{\Gamma}^\lambda{}_{\mu\nu}$ transforms inhomogeneously,

$$\widetilde{\Gamma}^\lambda{}_{\mu\nu} \to \frac{\partial y^\lambda}{\partial x^\alpha}\frac{\partial x^\beta}{\partial y^\mu}\frac{\partial x^\gamma}{\partial y^\nu}\widetilde{\Gamma}^\alpha{}_{\beta\gamma} + \frac{\partial y^\lambda}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial y^\mu \partial y^\nu}\,. \tag{32}$$

For these lecture notes, we will also introduce a notation for the Levi-Civita covariant derivative,

$$\widetilde{\nabla}_\mu T^\nu{}_\rho \equiv \partial_\mu T^\nu{}_\rho + \widetilde{\Gamma}^\nu{}_{\alpha\mu} T^\alpha{}_\rho - \widetilde{\Gamma}^\alpha{}_{\rho\mu} T^\nu{}_\alpha\,. \tag{33}$$

Why did we discuss all that torsion stuff? While the metricity condition can be rather easily motivated on physical grounds, the vanishing-torsion condition of general relativity has less rigorous physical foundations. In fact, in gauge approaches to gravitation it is assumed to be non-vanishing.

1.4 The curvature tensor

Spacetime curvature can be defined as the commutator of covariant derivatives,

$$[\nabla_\mu, \nabla_\nu] V^\rho = R_{\mu\nu}{}^\rho{}_\alpha V^\alpha + T_{\mu\nu}{}^\alpha \nabla_\alpha V^\rho\,. \tag{34}$$

Performing this computation explicitly does not take too long (it's 5-6 lines) **[Do it! ☺]** and we find the following definition for the *curvature tensor* $R_{\mu\nu}{}^\rho{}_\sigma$,

$$R_{\mu\nu}{}^\rho{}_\sigma \equiv \partial_\mu \Gamma^\rho{}_{\sigma\nu} - \partial_\nu \Gamma^\rho{}_{\sigma\mu} + \Gamma^\rho{}_{\alpha\mu}\Gamma^\alpha{}_{\sigma\nu} - \Gamma^\rho{}_{\alpha\nu}\Gamma^\alpha{}_{\sigma\mu}\,. \tag{35}$$

Contracting various indices of the curvature tensor with the metric, we can then construct the Ricci tensors $R_{\mu\nu}$ and the Ricci scalar R via

$$R_{\mu\nu} \equiv R_{\alpha\mu}{}^{\alpha}{}_{\nu}\,, \qquad R \equiv g^{\alpha\beta} R_{\alpha\beta}\,. \tag{36}$$

Note that in general $R_{\mu\nu} \neq R_{\nu\mu}$, that is, the Ricci tensor in presence of torsion is asymmetric. It is also useful to introduce the rank-(0,4) tensor $R_{\mu\nu\rho\sigma} = g_{\rho\alpha} R_{\mu\nu}{}^{\alpha}{}_{\sigma}$; this tensor has the symmetries

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}\,, \quad R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\sigma}\,, \quad R_{[\mu\nu\rho\sigma]} = 0\,. \tag{37}$$

Note that in general $R_{\mu\nu\rho\sigma} \neq R_{\rho\sigma\mu\nu}$ (incidentally, this is the reason that the Ricci tensor in presence of torsion is asymmetric). Moving on to the torsion-free covariant derivative, we may similarly define the commutator

$$[\widetilde{\nabla}_\mu, \widetilde{\nabla}_\nu] V^\rho = \widetilde{R}_{\mu\nu}{}^{\rho}{}_{\alpha} V^\alpha\,, \tag{38}$$

where now of course

$$\widetilde{R}_{\mu\nu}{}^{\rho}{}_{\sigma} \equiv \partial_\mu \widetilde{\Gamma}^{\rho}{}_{\sigma\nu} - \partial_\nu \widetilde{\Gamma}^{\rho}{}_{\sigma\mu} + \widetilde{\Gamma}^{\rho}{}_{\alpha\mu} \widetilde{\Gamma}^{\alpha}{}_{\sigma\nu} - \widetilde{\Gamma}^{\rho}{}_{\alpha\nu} \widetilde{\Gamma}^{\alpha}{}_{\sigma\mu}\,. \tag{39}$$

This tensor can also be used to construct a Ricci tensor $\widetilde{R}_{\mu\nu}$ and Ricci scalar $\widetilde{R}$ in a similar fashion,

$$\widetilde{R}_{\mu\nu} \equiv \widetilde{R}_{\alpha\mu}{}^{\alpha}{}_{\nu}\,, \qquad \widetilde{R} \equiv g^{\alpha\beta} \widetilde{R}_{\alpha\beta}\,. \tag{40}$$

Now we finally have $\widetilde{R}_{\mu\nu} = \widetilde{R}_{\nu\mu}$. This is inherited from the tensor $\widetilde{R}_{\mu\nu\rho\sigma} = g_{\rho\alpha} \widetilde{R}_{\mu\nu}{}^{\alpha}{}_{\sigma}$, which has the enhanced symmetries

$$\widetilde{R}_{\mu\nu\rho\sigma} = -\widetilde{R}_{\nu\mu\rho\sigma}\,, \quad \widetilde{R}_{\mu\nu\rho\sigma} = -\widetilde{R}_{\mu\nu\sigma\sigma}\,, \quad \widetilde{R}_{\mu\nu\rho\sigma} = \widetilde{R}_{\rho\sigma\mu\nu}\,, \quad \widetilde{R}_{\mu[\nu\rho\sigma]} = 0\,. \tag{41}$$

The third relation, sometimes called the "pair-commutation symmetry," guarantees that the induces Ricci tensors is symmetric. This is exactly what we encounter in general relativity, where torsion is set to zero.

1.5 The field equations of general relativity

There are many ways to derive the field equations of general relativity. Without going into details of variational principles in curved spacetime, a rather straightforward way to arrive at the field equations goes as follows.

1. In flat spacetime, energy-momentum is conserved, $\partial^\mu T_{\mu\nu} = 0$. In presence of curved spacetime, we hence expect $\widetilde{\nabla}^\mu T_{\mu\nu} = 0$.
2. If energy-momentum is to cause spacetime curvature, the only thing we can write down that involves the spacetime metric only up to second derivatives in a coordinate-invariant way is

$$a\widetilde{R}_{\mu\nu} + b\widetilde{R}g_{\mu\nu} + c\,g_{\mu\nu} = d\,T_{\mu\nu}\,. \tag{42}$$

3. We now impose that $\widetilde{\nabla}^\mu T_{\mu\nu} = 0$ still holds. This implies that $b = -a/2$. We can now absorb a into c and d and find the field equations of general relativity,

$$\boxed{\widetilde{R}_{\mu\nu} - \frac{1}{2}\widetilde{R}g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\,T_{\mu\nu}\,.} \tag{43}$$

Here, we inserted $8\pi G$ as the correct constant of proportionality, which can be derived in the weak-field limit via the Poisson equation, and we called $c/a = \Lambda$. Black hole solutions are vacuum solutions, and hence $T_{\mu\nu} = 0$ and $\Lambda = 0$. Taking the trace of (43) gives $\widetilde{R} = 0$ and for this reason the vacuum field equations are equivalent to

$$\widetilde{R}_{\mu\nu} = 0\,. \tag{44}$$

The constant Λ is called the cosmological constant, since in cosmology it gives rise to the expansion (or contraction) of the universe. In this lecture we are concerned with so-called asymptotically flat spacetimes and for this reason we set $\Lambda = 0$.

When reading textbooks on general relativity, we never see so many tildes. This is because I included the notion of torsion in this first chapter (and will keep doing so until the end of this chapter) for illustrative purposes. I think it is always helpful to see which results are robust across a whole range of gravitational theories, and which ones are specific to, say, general relativity. All that is to say, if you read a general relativity text and you find the field equations listed as $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G\,T_{\mu\nu}$, it coincides with what we mean by Eq. (43).

1.6 Killing equation

The *Killing equation* is simply the statement of an isometry, $\mathcal{L}_\xi g_{\mu\nu} = 0$. Given a metric $g_{\mu\nu}$, vector fields ξ^μ solving this equation are called *Killing vectors*. From Eq. (18) we already know that it takes the form

$$0 = \mathcal{L}_\xi g_{\mu\nu} = \xi^\alpha(\partial_\alpha g_{\mu\nu}) + (\partial_\mu \xi^\alpha) g_{\alpha\nu} + (\partial_\nu \xi^\alpha) g_{\mu\alpha} \,. \tag{45}$$

However, this is not the usual form we find for it in the literature. Recalling the definition of the covariant derivative, we can substitute

$$\partial_\alpha g_{\mu\nu} = \nabla_\alpha g_{\mu\nu} + \Gamma^\beta{}_{\mu\alpha} g_{\beta\nu} + \Gamma^\beta{}_{\nu\alpha} g_{\mu\beta} \,, \tag{46}$$

$$\partial_\mu \xi^\alpha = \nabla_\mu \xi^\alpha - \Gamma^\alpha{}_{\beta\mu} \xi^\beta \,, \tag{47}$$

$$\partial_\nu \xi^\alpha = \nabla_\nu \xi^\alpha - \Gamma^\alpha{}_{\beta\nu} \xi^\beta \,. \tag{48}$$

The Killing equation, after reshuffling and remembering $\nabla_\alpha g_{\mu\nu} = 0$ (because of the metricity condition) and $\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} = T_{\mu\nu}{}^\lambda$ (because in general, our connection has torsion), becomes

$$0 = \mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu + (T_\mu{}^\alpha{}_\nu + T_\nu{}^\alpha{}_\mu)\xi_\alpha = \tilde{\nabla}_\mu \xi_\nu + \tilde{\nabla}_\nu \xi_\mu \,. \tag{49}$$

The last expression is what we usually see in textbooks on general relativity (since therein, torsion vanishes), and it is also what we will compute in these lecture notes. To actually find Killing vectors it can be easier to work with the initial form of the Killing equation in terms of the Lie derivative, Eq. (45), but as we will see shortly, it is quite easy to prove that there are conservation laws for moving particles when we instead make use of the formulation (49).

1.7 Motion of test particles

Everything that has energy generates gravity, which in turn affects the spacetime geometry. This means that in general it is impossible to divide a gravitational system into separate parts. However, if a particle moving in such a system is small and its impact on the system is considered negligible, we may treat it as a test particle. And when we say "particle" we really mean massive and massless objects, even though in the context of this lecture we will exclusively work with massive particles.

It is useful to define the trajectory of such a particle as a function $x^\mu(\tau)$, where τ is called the *proper time*. Then it is possible to compute the particle's 4-velocity,

$$u^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \equiv \dot{u}^\mu \,. \tag{50}$$

We will often use the dot to indicate a derivative with respect to proper time. The 4-velocity u^μ is a vector, and we can always find a parametrization that normalizes its length. Massive particles move on causal, timelike curves (that is, with less than the speed of light) and in our signature convention this means that

$$g_{\mu\nu}u^\mu u^\nu = -1\,. \tag{51}$$

We can also define the particle's 4-momentum $p^\mu \equiv mu^\mu$, where m is the particle's rest mass. Then we have

$$g_{\mu\nu}p^\mu p^\nu = -m^2\,. \tag{52}$$

While for extended bodies the relation between 4-velocity and 4-momentum can be more complicated, for particles the difference is solely a rescaling by its mass. For the most part of these lectures, we will make use of the 4-velocity for simplicity (one less parameter to keep track of).

What we need now is a way to determine the particle's dynamics. Interestingly, there are several ways to arrive at an equation of motion for $x^\mu(\tau)$, and within general relativity those all end up being equivalent. *Geodesics* are defined as curves that extremize their proper path length. Within the variational principle, we can consider the action

$$S[x^\mu(\tau)] = \int \mathrm{d}\tau = \int \mathrm{d}\tau \sqrt{-g_{\mu\nu}u^\mu u^\nu}\,. \tag{53}$$

The variational principle then gives (after $\sim$ 1 page of algebra) the *geodesic equation*

$$\boxed{\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \widetilde{\Gamma}^\mu{}_{\alpha\beta}\, u^\alpha u^\beta = 0\,.} \tag{54}$$

Notice that here only the Levi-Civita connection $\widetilde{\Gamma}^\lambda{}_{\mu\nu}$ makes an appearance. This is because the length of a curve is a purely metric concept, and hence only the part of the connection directly specified by the metric makes an appearance.

However, it is also possible to define a group of curves by requiring that they parallel transport their own tangent vector. This gives rise to the *autoparallel equation*

$$\boxed{u^\alpha \nabla_\alpha u^\mu = 0\,.} \tag{55}$$

And now, the big question: are those equations equivalent? In general, the answer is no. One way to see that is to recast the geodesic equation in terms of covariant derivatives. Expanding the autoparallel equation explicitly, we find

$$0 = u^\alpha(\partial_\alpha u^\mu) + u^\alpha \Gamma^\mu{}_{\beta\alpha} u^\beta = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\partial u^\mu}{\partial x^\alpha} + \Gamma^\mu{}_{\alpha\beta} u^\alpha u^\beta = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu{}_{\alpha\beta} u^\alpha u^\beta \,, \tag{56}$$

Where the last equal sign follows simply from the chain rule. We thereby arrive at an alternative version of the autoparallel equation,

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu{}_{\alpha\beta}\, u^\alpha u^\beta = 0\,. \tag{57}$$

This looks almost identical to Eq. (54), except that this carries the full connection $\Gamma^\lambda{}_{\mu\nu}$ instead of merely the Levi-Civita part $\widetilde{\Gamma}^\lambda{}_{\mu\nu}$. To capture the difference, we can insert $\Gamma^\lambda{}_{\mu\nu} = \widetilde{\Gamma}^\lambda{}_{\mu\nu} + K^\lambda{}_{\mu\nu}$ and find

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \left(\widetilde{\Gamma}^\mu{}_{\alpha\beta} + T_\alpha{}^\mu{}_\beta\right) u^\alpha u^\beta = 0\,, \tag{58}$$

Conversely, we may also impose the torsion-free autoparallel condition $u^\mu \widetilde{\nabla}_\mu u^\nu = 0$. For the reasons stated above, this implies the geodesic equation directly:

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \widetilde{\Gamma}^\mu{}_{\alpha\beta}\, u^\alpha u^\beta = 0\,. \tag{59}$$

Okay, so depending on what we stipulate we can have different motion for point particles. This may sound confusing, but it is really a coincidence that these two notions do coincide in general relativity. That being said, one may wonder if both equations are compatible with $g_{\alpha\beta}u^\alpha u^\beta = -1$. Taking the directional derivative of this scalar along the geodesic, one finds

$$0 = u^\alpha \nabla_\alpha \left(g_{\beta\gamma} u^\beta u^\gamma\right) = u^\alpha g_{\beta\gamma}\left[(\nabla_\alpha u^\beta) u^\gamma + (\nabla_\alpha u^\gamma) u^\beta\right] = u_\beta u^\alpha (\nabla_\alpha u^\beta) \tag{60}$$

$$= u_\beta u^\alpha \left(\widetilde{\nabla}_\alpha u^\beta + K^\beta{}_{\gamma\alpha} u^\gamma\right) = u_\beta u^\alpha \widetilde{\nabla}_\alpha u^\beta + u^\alpha u^\beta u^\gamma K_{\alpha\beta\gamma} = u_\beta u^\alpha \widetilde{\nabla}_\alpha u^\beta\,. \tag{61}$$

What does all of that mean? No matter if we use the geodesic equation (54) or the autoparallel equation (55), the expression $g_{\alpha\beta}u^\alpha u^\beta$ is always constant along the curve. In other words: we have found our first constant of motion!

And speaking of constants of motion: since we already mentioned Killing vectors, let's see how those can give rise to extra conserved quantities along geodesics! Unlike the previous section, we will consider first the torsion-free case (since this is the case relevant for the following lectures), but we will briefly mention afterwards what changes when there is non-vanishing torsion.

Say we have a vector k^μ that satisfies the torsion-free Killing equation

$$\widetilde{\nabla}_\mu k_\nu + \widetilde{\nabla}_\nu k_\mu = 0\,. \tag{62}$$

Next, say that we have a particle that satisfies the geodesic equation which we will write here in its covariant-derivative form,

$$u^\alpha \widetilde{\nabla}_\alpha u^\mu = 0\,. \tag{63}$$

Then, the quantity $Q = g_{\alpha\beta} u^\alpha k^\beta$ is conserved along the geodesic! To prove it, we calculate its derivative along the curve,

$$u^\gamma \widetilde{\nabla}_\gamma Q = u^\gamma \widetilde{\nabla}_\gamma (g_{\alpha\beta} u^\alpha k^\beta) = k_\alpha u^\gamma (\widetilde{\nabla}_\gamma u^\alpha) + u^\gamma u^\beta (\widetilde{\nabla}_\gamma k_\beta) = 0 + \frac{1}{2} u^\gamma u^\beta (\widetilde{\nabla}_\gamma k_\beta + \widetilde{\nabla}_\beta k_\gamma) = 0\,. \tag{64}$$

In the last two equalities we made use of the geodesic equation and the Killing equation. Wonderful, we have a conserved quantity! Does it also work like this in the case of non-vanishing torsion? Unfortunately, the answer is *no*. Killing vectors in the presence of torsion do not give rise to conserved quantities. However, we can define something similar. Consider the equation

$$\nabla_\mu k_\nu + \nabla_\nu k_\mu = 0\,. \tag{65}$$

This is, most decisively, *not* a Killing equation. However, it implies that the quantity $Q = g_{\alpha\beta} k^\alpha u^\beta$ is conserved along autoparallels, which are defined by $u^\alpha \nabla_\alpha u^\mu = 0$. The steps to prove this are identical to the prove above, just without the tildes for decoration. To practice these sort of calculations, you can try to derive it yourself. **[Do it! ☺]**

So where does this leave us? We have seen that in general relativity, massive particles follow geodesics or autoparallels—they are the same. Moreover, for each Killing vector we can define a quantity that is conserved under geodesics or autoparallel motion. This will come in handy.

In settings with spacetime torsion, however, geodesics and autoparallels do *not* coincide. Moreover, Killing vectors in such settings do not give rise to conserved quantities. For autoparallel motion, however, we can define something similar to Killing vectors that leads to conserved quantities.

That being said, nobody stops us from taking the geodesic equation as the fundamental particle equation of motion regardless of whether there is or isn't torsion. In that case, we can just take a Killing vector and show that it leads to a conserved quantity under $u^\alpha \widetilde{\nabla}_\alpha$, that is, under geodesic motion. What is realized in nature? While most physicists nowadays believe that the geodesic equation is the correct description, ultimately only Nature will be able to tell us what is realized in our Universe.

Chapter 2

Geodesic motion around the Schwarzschild black hole

While realistic, astrophysical black holes are time-dependent, spinning entities surrounded by accreting matter and strong electromagnetic fields, muck about black holes can be learned from a symmetry-enhanced scenario. Assuming axial symmetry around the axis of rotation, as well as vacuum, leads to the Kerr black hole. Neglecting the black holes' angular momentum allows one to focus instead on a static, spherically symmetric configuration. While angular momentum plays an important role in the astrophysics and gravitational wave properties of black holes, we will neglect it in these lecture notes for simplicity.

In this Chapter, we will first derive the Schwarzschild metric: an exact, spherically symmetric vacuum solution of the field equations of general relativity. In fact, the Birkhoff theorem states that it is the unique solution, labeled merely by the parameter M which denotes the mass of the black hole. In the limit of $M \to 0$ one recovers flat spacetime of special relativity.

2.1 Spherically symmetric vacuum solution

Let us start with the following parametrization of a static, spherically symmetric metric:

$$\mathrm{d}s^2 = -A(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{B(r)} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2\,. \tag{66}$$

Here we employ spherical coordinates $x^\mu = \{t, r, \theta, \varphi\}$, and in what follows we will abbreviate $A(r) \equiv A$ and $B(r) \equiv B$ for brevity. The metric components and inverse metric components are

$$g_{tt} = -A\,, \quad g_{rr} = \frac{1}{B}\,, \quad g_{\theta\theta} = r^2\,, \quad g_{\varphi\varphi} = r^2 \sin^2\theta\,, \tag{67}$$

$$g^{tt} = -\frac{1}{A}\,, \quad g^{rr} = B\,, \quad g^{\theta\theta} = \frac{1}{r^2}\,, \quad g^{\varphi\varphi} = \frac{1}{r^2\sin^2\theta}\,. \tag{68}$$

The Christoffel symbols are

$$\begin{aligned} &\tilde{\Gamma}^t{}_{tr} = \frac{A'}{2A}\,, \quad \tilde{\Gamma}^r{}_{tt} = \frac{BA'}{2}\,, \quad \tilde{\Gamma}^r{}_{rr} = -\frac{B'}{2B}\,, \quad \tilde{\Gamma}^r{}_{\theta\theta} = -Br\,, \quad \tilde{\Gamma}^r{}_{\varphi\varphi} = -Br\sin^2\theta\,, \\ &\tilde{\Gamma}^\theta{}_{r\theta} = \tilde{\Gamma}^\varphi{}_{r\varphi} = \frac{1}{r}\,, \quad \tilde{\Gamma}^\theta{}_{\varphi\varphi} = -\sin\theta\cos\theta\,, \quad \tilde{\Gamma}^\varphi{}_{\theta\varphi} = \frac{\cos\theta}{\sin\theta}\,. \end{aligned} \tag{69}$$

But *why* is this a good ansatz to make for what we are after (a static, spherically symmetric vacuum solution)? By that we don't mean the specific choice of coordinates, because we can always change those, but rather the form of the metric. To see why this makes sense, we can check the isometries of the above metric, and to study the isometries we need to look at the Killing vectors. The Killing equation takes the form

$$0 = \partial_\mu k_\nu + \partial_\nu k_\mu - 2\widetilde{\Gamma}^\alpha{}_{\mu\nu} k_\alpha\,. \tag{70}$$

This equation is a tensorial equation with two indices, and you can show that is symmetric in exchange of $\mu \leftrightarrow \nu$. **[Do it! ☺]** The ten components of this equation are given by the following combination of indices, $(\mu\nu = tt, tr, t\theta, t\varphi, rr, r\theta, r\varphi, \theta\theta, \theta\varphi, \varphi\varphi)$:

$$0 = g_{tt}\partial_t k^t - \widetilde{\Gamma}^r{}_{tt} g_{rr} k^r\,, \tag{71}$$

$$0 = g_{rr}\partial_t k^r + \partial_r g_{tt} k^t - 2 g_{tt}\widetilde{\Gamma}^t{}_{tr} k^t\,, \tag{72}$$

$$0 = g_{\theta\theta}\partial_t k^\theta + g_{tt}\partial_\theta k^t\,, \tag{73}$$

$$0 = g_{\varphi\varphi}\partial_t k^\varphi + g_{tt}\partial_\varphi k^t\,, \tag{74}$$

$$0 = \partial_r g_{rr} k^r - g_{rr}\widetilde{\Gamma}^r{}_{rr} k^r\,, \tag{75}$$

$$0 = \partial_r g_{\theta\theta} k^\theta + g_{rr}\partial_\theta k^r - 2 g_{\theta\theta}\widetilde{\Gamma}^\theta{}_{r\theta} k^\theta\,, \tag{76}$$

$$0 = \partial_r g_{\varphi\varphi} k^\varphi + g_{rr}\partial_\varphi k^r - 2 g_{\varphi\varphi}\widetilde{\Gamma}^\varphi{}_{r\varphi} k^\varphi\,, \tag{77}$$

$$0 = g_{\theta\theta}\partial_\theta k^\theta - g_{rr}\widetilde{\Gamma}^r{}_{\theta\theta} k^r\,, \tag{78}$$

$$0 = \partial_\theta g_{\varphi\varphi} k^\varphi + g_{\theta\theta}\partial_\varphi k^\theta - 2 g_{\varphi\varphi}\widetilde{\Gamma}^\varphi{}_{\theta\varphi} k^\varphi\,, \tag{79}$$

$$0 = g_{\varphi\varphi}\partial_\varphi k^\varphi - g_{rr}\widetilde{\Gamma}^r{}_{\varphi\varphi} k^r - g_{\theta\theta}\widetilde{\Gamma}^\theta{}_{\varphi\varphi} k^\theta\,. \tag{80}$$

Using these equations we can prove that the metric (66) is static, which really just means that the vector $\xi = \partial_t$ is a Killing vector, which can be verified with the above relations upon inserting $\xi^t = 1$ with all other components vanishing. **[Do it! ☺]** The metric is also spherically symmetric, since it admits the three Killing vectors

$$\rho_1 = \partial_\varphi \,, \tag{81}$$

$$\rho_2 = \sin\varphi\, \partial_\theta + \cot\theta \cos\varphi\, \partial_\varphi \,, \tag{82}$$

$$\rho_3 = -\cos\varphi\, \partial_\theta + \cot\theta \sin\varphi\, \partial_\varphi \,. \tag{83}$$

These vectors are spacelike, not orthogonal, but span the $SO(3)$ Lie algebra

$$[\rho_A, \rho_B] = -\epsilon_{ABC}\rho_C \,, \tag{84}$$

with $\epsilon_{123} \equiv 1$. This can be shown by using the definition of the vector commutator defined in Sec. 1.2, and it does not take a long time to work it out (about 10 lines in total) **[Do it! ☺]**

Having convinced ourselves that the ansatz (66) is justified for a spherically symmetric static geometry, we can now insert this metric into the vacuum field equations of general relativity. In principle it is straightforward to first compute the Christoffel symbols $\widetilde{\Gamma}^\lambda{}_{\mu\nu}$, and from there the Riemann tensor $\widetilde{R}^\mu{}_{\nu\rho\sigma}$ and its contractions (Ricci tensor $\widetilde{R}_{\mu\nu}$ and Ricci scalat $\widetilde{R}$), but in practice it can be quite cumbersome to do this by hand, especially when you are just taking a class for a few days and do not want to get overly invested.

For this reason, it is a good idea to use computer algebra. In appendix A you can hence find a list of download links to Mathematica worksheets that do these computations for you. Since this lecture is not focused on computer algebra we will not go into excruciating detail, but from the worksheet you can get a good idea of how to use it—it does not make use of any library and it is rather easy to add additional functionality, if necessary, if you have previously been exposed to Mathematica. At any rate, in these lecture notes we will use the worksheet outputs to arrive at the components of the relevant tensors and quote their results here.

For the above ansatz, one finds the following expressions:

$$\widetilde{R}_{tt} = \frac{4ABA' - rBA'^2 + rAA'B' + 2rABA''}{4rA} \,, \tag{85}$$

$$\widetilde{R}_{rr} = \frac{rBA'^2 - 4A^2B' - rAA'B' - 2rABA''}{4rA^2B} \,, \tag{86}$$

$$\widetilde{R}_{\theta\theta} = \frac{2A - 2AB - rBA' - rAB'}{2A} \,, \tag{87}$$

$$\widetilde{R}_{\varphi\varphi} = \sin^2\theta \widetilde{R}_{\theta\theta} \,. \tag{88}$$

The combination $r(\widetilde{R}_{tt} + AB\widetilde{R}_{rr})$ gives $AB' - BA' = 0$, which implies

$$\frac{A}{B} = k_1\,, \tag{89}$$

where k_1 is a constant. We can absorb this constant into a redefinition of the time coordinate $t \to t/\sqrt{k_1}$ and hence are free to set $k_1 = 1$. Substituting this into the $\theta\theta$-component gives

$$1 - A - rA' = 0\,, \tag{90}$$

which is solved by

$$A = 1 + \frac{k_2}{r}\,, \tag{91}$$

where k_2 is a constant of dimension length. All equations are now solved. For $k_2 = 0$ we arrive at flat spacetime, so it is already plausible to assume that $k_2 \propto GM$, where M is the black hole mass, since for $M \to 0$ we assume to return to flat spacetime. The Newtonian limit (see in Sec. 2.2) then fixes the constant of proportionality to a factor of -2 such that $k_2 = -2GM$. The result,

$$A = B = 1 - \frac{2GM}{r}\,, \tag{92}$$

solves all remaining equations. Calling $A(r) = B(r) = f(r)$ we arrive at the standard form of the Schwarzschild metric,

$$\boxed{\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2\,, \quad f(r) = 1 - \frac{2GM}{r}\,.} \tag{93}$$

One can prove that this metric is the unique spherically symmetric and static vacuum solution of the field equations of general relativity (which is called Birkhoff's theorem).

2.2 Motion of test particles via the geodesic equation

One might be tempted to start with the Christoffel symbols and write down the equation of motion. This is possible, since we computed the form of the non-vanishing Christoffel symbols previously. They are

$$\tilde{\Gamma}^t{}_{tr} = -\tilde{\Gamma}^r{}_{rr} = \frac{f'}{2f}\,, \quad \tilde{\Gamma}^r{}_{tt} = \frac{ff'}{2}\,, \quad \tilde{\Gamma}^r{}_{\theta\theta} = -fr\,, \quad \tilde{\Gamma}^r{}_{\varphi\varphi} = -fr\sin^2\theta\,, \tag{94}$$

$$\tilde{\Gamma}^{\theta}{}_{r\theta} = \tilde{\Gamma}^{\varphi}{}_{r\varphi} = \frac{1}{r}\,, \quad \tilde{\Gamma}^{\theta}{}_{\varphi\varphi} = -\sin\theta\cos\theta\,, \quad \tilde{\Gamma}^{\varphi}{}_{\theta\varphi} = \frac{\cos\theta}{\sin\theta}\,. \tag{95}$$

Inserting the function $f(r) = 1 - 2GM/r$ one finds explicitly

$$\tilde{\Gamma}^{t}{}_{tr} = -\tilde{\Gamma}^{r}{}_{rr} = \frac{GM}{r(r-2GM)}\,, \quad \tilde{\Gamma}^{r}{}_{tt} = \left(1 - \frac{2GM}{r}\right)\frac{GM}{r^2}\,, \tag{96}$$

$$\tilde{\Gamma}^{r}{}_{\theta\theta} = 2GM - r\,, \quad \tilde{\Gamma}^{r}{}_{\varphi\varphi} = (2GM - r)\sin^2\theta\,, \tag{97}$$

$$\tilde{\Gamma}^{\theta}{}_{r\theta} = \tilde{\Gamma}^{\varphi}{}_{r\varphi} = \frac{1}{r}\,, \quad \tilde{\Gamma}^{\theta}{}_{\varphi\varphi} = -\sin\theta\cos\theta\,, \quad \tilde{\Gamma}^{\varphi}{}_{\theta\varphi} = \frac{\cos\theta}{\sin\theta}\,. \tag{98}$$

The geodesic equation is

$$0 = \ddot{t} + 2\tilde{\Gamma}^{t}{}_{tr}\dot{t}\dot{r}\,, \tag{99}$$

$$0 = \ddot{r} + \tilde{\Gamma}^{r}{}_{tt}\dot{t}^2 + \tilde{\Gamma}^{r}{}_{rr}\dot{r}^2 + \tilde{\Gamma}^{r}{}_{\theta\theta}\dot{\theta}^2 + \tilde{\Gamma}^{r}{}_{\varphi\varphi}\dot{\varphi}^2\,, \tag{100}$$

$$0 = \ddot{\theta} + 2\tilde{\Gamma}^{\theta}{}_{r\theta}\dot{r}\dot{\theta} + \tilde{\Gamma}^{\theta}{}_{\varphi\varphi}\dot{\varphi}^2\,, \tag{101}$$

$$0 = \ddot{\varphi} + 2\tilde{\Gamma}^{\varphi}{}_{r\varphi}\dot{r}\dot{\varphi} + 2\tilde{\Gamma}^{\varphi}{}_{\theta\varphi}\dot{\theta}\dot{\varphi}\,. \tag{102}$$

This is an ordinary, second-order partial differential equation, and can be solved numerically rather easily provided suitable boundary conditions are provided. We can (try to) read off some properties:

- Motion in the equatorial plane ($\theta = \pi/2, \dot{\theta} = 0$) is allowed.
- If $\dot{\varphi}$ is constant, the equation for $\ddot{r}$ seems to imply unbound motion, since it contains $\ddot{r} - \dot{\varphi}^2 r = 0$, solved by $r \sim \exp[\pm\text{const}\tau]$. But it is not clear if $\dot{\varphi}$ is a constant or not.
- And, very roughly speaking, for Newtonian radial motion we can assume $\ddot{t} \approx 1$ and $\dot{r}^2 \ll 1$ (non-relativistic motion) and $r \gg 2GM$ (Newtonian gravity is valid), as well as $\dot{\theta} = \dot{\varphi} = 0$ (radial motion). Under such assumptions, the second equation becomes $\ddot{r} = -\tilde{\Gamma}^{t}{}_{rr} \approx -GM/r^2$, which is Newton's law of gravitation in the radial coordinate. This last step is very helpful in identifying the integration constant k_2 of Sec. 2.1.

Of course we can study this equation more on a purely numerical level, but it makes sense to go beyond that and make use of the conserved quantities we derived earlier. This way, as we will see shortly, we can reduce the equations of motion to first order and qualitatively discuss bound particle motion without ever solving differential equations.

2.3 Motion of test particles via the constants of motion

As pointed out, we can do one better. As we may have guessed from the presence of four Killing vectors, there are four conserved quantities for geodesic motion in the Schwarzschild geometry.

We can define the following four constants (the minus sign in the first equation is conventional):

$$E = -g_{\mu\nu} u^\mu \xi^\nu = \left(1 - \frac{2GM}{r}\right) \dot{t}\,, \tag{103}$$

$$L_1 = g_{\mu\nu} u^\mu \rho_1^\nu = r^2 \sin^2\theta\, \dot{\varphi}\,, \tag{104}$$

$$L_2 = g_{\mu\nu} u^\mu \rho_2^\nu = +r^2 \sin\varphi\, \dot{\theta} + r^2 \sin\theta \cos\theta \cos\varphi\, \dot{\varphi}\,, \tag{105}$$

$$L_3 = g_{\mu\nu} u^\mu \rho_2^\nu = -r^2 \cos\varphi\, \dot{\theta} + r^2 \sin\theta \cos\theta \sin\varphi\, \dot{\varphi}\,. \tag{106}$$

The three constants L_1, L_2, and L_3 correspond to the three independently conserved components of angular momentum. However, as we saw in the previous section, due to spherical symmetry we may consider equatorial orbits without loss of generality. Alternatively, for any given conserved angular momentum, we may orient the coordinate system in such a way that its axis corresponds to $\theta = 0$. Either way, the motion happens in the $\theta = \pi/2$ plane at $\dot{\theta} = 0$. This then forces the two constants L_2 and L_3 to become trivial, and we finally arrive at a set of two constants of motion,

$$E = \left(1 - \frac{2GM}{r}\right) \dot{t}\,, \qquad L \equiv L_1 = r^2\, \dot{\varphi}\,, \quad L_2 = L_3 = 0\,. \tag{107}$$

But recall that $g_{\mu\nu} p^\mu p^\nu = -1$ is of course also a constant of motion—we will get back to that in a second, since that is the key to arrive at a simpler equation of motion! For now, though, let us point out that we work here with the 4-velocity instead of the 4-momentum p^μ. This is OK since for point particles of mass m we have $p^\mu = m u^\mu$. It also shows that the physical interpretation of E is not the energy of the particle, but the energy per rest mass; similarly, L is the angular momentum per rest mass. Accordingly, E is dimensionless, while L has units of length-squared.

But aside from a rescaling by the rest mass, what are reasonable values for those parameters? A particle at rest at spatial infinity ($r = \infty$) has a 4-velocity that is proportional to the timelike Killing vector, $u^\mu = (1, 0, 0, 0)$ which implies that $\dot{t} = 1$. This, in turn, implies that $E = 1$ for such a particle. If its radial velocity is non-vanishing, we still have that $-1 = -\dot{t}^2 + \dot{r}^2$ at infinity, which implies that $|\dot{r}| > 0$ increases the value of $\dot{t}$ since $\dot{t}^2 = 1 + \dot{r}^2$. What this means is that E is bounded from below by 1 for all particles that come from infinity. This makes sense: Restoring the mass and the speed of light for a second, we know that the energy of a particle (in flat spacetime!) has to be at least mc^2, and $E = 1$ is precisely that statement.

Okay, now it is finally time to remember that the normalization condition of the 4-velocity of a massive particle is also a conserved quantity. Writing

$$u^\mu = (t(\tau), r(\tau), \theta(\tau), \varphi(\tau)) \,, \tag{108}$$

we find explicitly

$$-1 = g_{\mu\nu} u^\mu u^\nu = -f\dot{t}^2 + \frac{\dot{r}^2}{f} + r^2\dot{\theta}^2 + r^2 \sin^2\theta \dot{\varphi}^2 \,. \tag{109}$$

From now on we no longer write $t(\tau)$, $r(\tau)$, and so on, but omit the τ-dependence for notational brevity and just write t and r, and so on, with the implicit assumption that these are all functions of the proper time τ. As a next step, we substitute the constants of motion from Eq. (107) and find

$$-1 = -f\frac{E^2}{f^2} + \frac{\dot{r}^2}{f} + r^2\dot{\theta}^2 + r^2 \sin^2\theta \frac{L^2}{r^4} \,. \tag{110}$$

Since the Schwarzschild metric is spherically symmetric, we can set $\theta = \pi/2$ without loss of generality; this also follows from the geodesic equation we discussed earlier. Then, collecting some common factors and rearranging some terms, the above relation becomes

$$\dot{r}^2 = E^2 - V_{\text{eff}}(r) \,, \quad V_{\text{eff}}(r) = \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{L^2}{r^2}\right) \,. \tag{111}$$

As the name suggests, V_{eff} is called the *effective potential* since it governs the particle motion. Since it is the potential for a particle with mass 1 (remember? we normalized everything to the particle's rest mass m), the effective potential is a dimensionless quantity. Its shape as a function of radius r depends solely on the angular momentum parameter L and the black hole mass M. It may seem a bit odd, but Eq. (111)—together with the definition of E and L via Eq. (107)—is our equation of motion, and it is all we need to describe particle motion inside the Schwarzschild geometry (for $r > 2GM$, that is). Okay, but how do we describe particle motion qualitatively?

2.4 Bound orbits for massive particles

To qualitatively describe the orbits, the only think we need to remember is that the left-hand side of Eq. (111) is positive. This means that $V_{\text{eff}}(r) \leq E^2$ is strictly required. When we plot this graphically, given a particle energy E, we see that the allowed regions of motion for the radial variable r are those where E^2 is above the potential line; see also Fig. 1. As it turns out, for very large values of E only unbound orbits are possible, whereas other values allow for bound orbits.

For the context of the present lecture, bound orbits are of primary importance. Studying the effective potential a bit more, it helps to isolate its extrema. We find

$$V'_{\text{eff}}(r_\pm) = 0\,, \quad r_\pm = \frac{L}{2GM}\left(L \pm \sqrt{L^2 - 12(GM)^2}\right)\,, \tag{112}$$

that is, for $L > 2\sqrt{3}GM \approx 3.46GM$ there exist two extrema; the maximum is at $r = r_-$ and the minimum at $r = r_+$. A bound orbit is possible only if the potential has a maximum (in other words: there is a potential barrier), since otherwise nothing would stop the particle from falling beyond $r = 2GM$ and never to return outside of the black hole. Such a maximum exists only if the angular momentum is not too small (since otherwise the black hole always wins). Intuitively, this makes sense.

However, a large angular momentum is only a necessary requirement. Another necessary requirement is $E^2 < 1$ since otherwise the particle can escape to $r \to \infty$. And last, we also need to make sure that the particle does not plunge into the black hole, hence $E^2 < V_{\text{eff}}(r_-)$. Collecting these conditions, we find

$$L > 2\sqrt{3}GM\,, \quad E^2 < 1\,, \quad E^2 < V_{\text{eff}}(r_-) = \frac{2[4(GM)^2 - L(L - \sqrt{L^2 - 12(GM)^2})]^2}{L(L - \sqrt{L^2 - 12(GM)^2})^3}\,. \tag{113}$$

This last condition is what we are after: given the initial values of E and L we can now check easily if they correspond to a bound orbit or not.

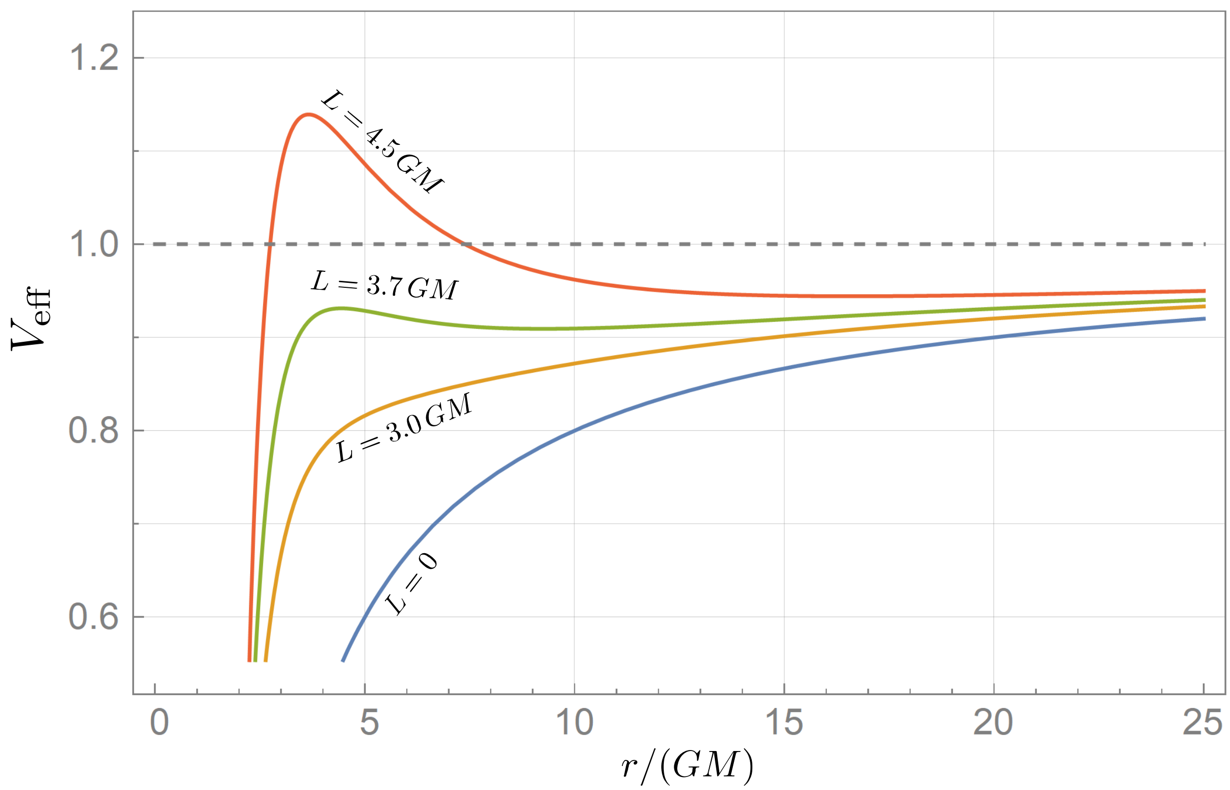

Figure 1: Effective potential $V_{\rm eff}(r)$ as a function of dimensionless distance $r/(GM)$ for the Schwarzschild metric, for different angular momenta L. The dashed line corresponds to the asymptotic value of $V_{\rm eff}(r \to \infty)$. Bound orbits only exist if (i) there is a potential maximum (to create an inner turning point), (ii) the quantity $E^2 < 1$ (to avoid particles to escape to $r = \infty$), and (iii) the quantity E^2 is less than the potential maximum (to avoid plunging orbits).

Chapter 3

Gravitational waves: basic notions

The difficulty inherent to general relativity (and modified gravity theories) lies in the non-linearity of the field equations, making it rather difficult to construct exact solutions. Basically, it is difficult to define isolated parts of the gravitational system, since everything couples to everything. A gravitational wave deforms a black hole, which responds by tidal deformation, emitting other gravitational waves, and so on.

So what can we do? There are many different approaches, including numerical relativity and systematic perturbation theory. Since in these lectures we care about leading-order effects in binary systems with a large mass ratio, we will focus on the perturbative approach. What's more, we will consider linearized gravity around flat spacetime. This is seemingly unjustified, since the gravitational field in the vicinity of black holes can become rather large, and gravitational waves can scatter off of spacetime curvature, but this method still captures the leading-order effects. For example, the orbital decay in the Hulse–Taylor pulsar (which consists of two neutron stars orbiting each other at a large distance) can be correctly described, within a few percent, with formulas that we will derive in this section. The discussion in Sec. 3.2 is partly based on Chs. 17 & 18 of the excellent book by Hobson *et al.* [3].

3.1 Lightning review of linearized gravity

To begin with, we split spacetime into that flat background $\eta_{\mu\nu}$ and a gravitational perturbation $h_{\mu\nu} = h_{\nu\mu}$ that is assumed very small, $2\kappa h_{\mu\nu} \ll 1$, such that

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu} \,. \tag{114}$$

Here, the factor $\kappa = \sqrt{8\pi G}$ is conventional and inserted so that the field $h_{\mu\nu}$ has dimensions of energy (in units where $\hbar = c = 1$). The field $h_{\mu\nu}$ (or, rather, its two massless modes of spin 2) is called the graviton, when the theory is quantized. Here we will perform classical field theory, but we can still think of $h_{\mu\nu}$ as the gravitational ripples in spacetime (modulo gauge-dependent expressions—we will get back to that later).

What we can do now is expand everything in terms of κ and only keep leading-order terms. The theory we then arrive at is called linearized gravity, and we can then use the method of Green functions and partial differential equations in flat spacetime to compute the gravitational wave signatures stemming from the motion of particles.

To first order in κ, the inverse metric then is

$$g^{\mu\nu} = \eta^{\mu\nu} - 2\kappa h^{\mu\nu} + \mathcal{O}(\kappa^2)\,, \tag{115}$$

where we raise and lower indices on $h_{\mu\nu}$ with $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$, respectively. For example, in a slight abuse of notation, we denote $h^{\mu}_{\nu} \equiv \eta_{\nu\alpha} h^{\mu\alpha}$. The determinant of the metric is expanded as

$$\sqrt{-g} = 1 + \kappa h + \mathcal{O}(\kappa^2)\,, \tag{116}$$

where we defined h, the trace of the metric perturbation, as follows:

$$h \equiv \eta^{\mu\nu} h_{\mu\nu}\,. \tag{117}$$

Next, to obtain the field equations of linearized gravity, we need to expand the Christoffel symbols and the curvature tensors. To first order, the Christoffel symbols then take the form

$$\Gamma^{\lambda}{}_{\mu\nu} = \kappa(\partial_\mu h^{\lambda}_{\nu} + \partial_\nu h^{\lambda}_{\mu} - \partial^{\lambda} h_{\mu\nu}) + \mathcal{O}(\kappa^2)\,. \tag{118}$$

This then gives rise to the Ricci tensor and scalar, which, to first order, turn out to be

$$R_{\mu\nu} = -\kappa\left(\Box h_{\mu\nu} + \partial_\mu\partial_\nu h - \partial_\mu\partial^\alpha h_{\alpha\nu} - \partial_\nu\partial^\alpha h_{\mu\alpha}\right)\,, \tag{119}$$

$$R = -2\kappa(\Box h - \partial^\mu\partial^\nu h_{\mu\nu}) + \mathcal{O}(\kappa^2)\,. \tag{120}$$

The left-hand side of the field equations then are

$$R_{\mu\nu} - \frac{1}{2} R\eta_{\mu\nu} = \kappa\left(-\Box h_{\mu\nu} - \partial_\mu\partial_\nu h + \partial_\mu\partial^\alpha h_{\alpha\nu} + \partial_\nu\partial^\alpha h_{\alpha\mu} + \Box h\eta_{\mu\nu} - \eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta}\right)\,. \tag{121}$$

Now, without going into too much detail, the field $h_{\mu\nu}$ is only unique up to a gauge transformation,

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \,, \tag{122}$$

where $2\kappa\epsilon^\mu \ll 1$ is an infinitesimal vector field. This gauge invariance is what survives from the diffeomorphism invariance of general relativity in the linearized limit. This is not too surprising, and you may actually already know something like this from electromagnetism, wherein the gauge field A_μ has a gauge invariance $A_\mu \to A_\mu + \partial_\mu \omega$ for some function ω. Since the "graviton" is also a gauge boson (much like the "photon" A_μ), this similarity of structures is to be expected and should not come as a total surprise.

However, the physics should not depend on the gauge! That is, if we use $h_{\mu\nu}$ to do our computations, or $h_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu$, the end result should not depend on it. What is the end result? One observable in the gravitational field is the curvature (that is, the tidal force between two points). The curvature should be gauge-independent. And, since the Lagrangian of general relativity is a scalar made up of a curvature invariant, the Lagrangian itself as well as the field equations should also be gauge-invariant. (In analogy to electromagnetism, where the "curvature" is $F_{\mu\nu}$, the electromagnetic field strength, which is also gauge-invariant.) For example, you can check that $R_{\mu\nu}$ and R are, in fact, gauge-invariant. **[Do it! ☺]**

Now, what to do with the gauge invariance? To do calculations in the real world, it is helpful to *fix* the gauge. To that end, we can demand some condition on the $h_{\mu\nu}$'s to hold, similar to the good old Lorenz gauge in electromagnetism (remember, when we demanded that, for example, $\partial^\mu A_\mu = 0$). Similarly, we demand that

$$\partial^\mu h_{\mu\nu} \overset{*}{=} \lambda \partial_\nu h \,. \tag{123}$$

One can show **[Do it! ☺]** that there exists a gauge transformation ϵ_μ that guarantees this transformation, provided $\lambda \neq 1$. If we take the above relation, and insert it into the field equations, we end up with

$$R_{\mu\nu} - \frac{1}{2} R \eta_{\mu\nu} \overset{*}{=} \kappa \left[-\Box \hat{h}_{\mu\nu} + \frac{1-2\lambda}{1-\lambda d} \left(\eta_{\mu\nu} \Box \hat{h} - \partial_\mu \partial_\nu \hat{h} \right) \right] . \tag{124}$$

From here, it is convenient to set $\lambda = 1/2$. In that gauge, the field equations of linearized general relativity take the form

$$\Box \hat{h}_{\mu\nu} \overset{*}{=} -\kappa T_{\mu\nu} \,, \quad g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu} \,, \quad \hat{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu} \,, \quad \kappa = \sqrt{8\pi G} \,. \tag{125}$$

In the textbook literature on linearized general relativity we often find a slightly different form, related to a different parametrization of the gravitational field perturbation, in which case

$$\Box \hat{h}_{\mu\nu} \stackrel{*}{=} -2\kappa^2 T_{\mu\nu}\,, \quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \quad \hat{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}\,, \quad \kappa = \sqrt{8\pi G}\,. \tag{126}$$

Note that here is no factor of 2κ in the metric perturbation. Since this formulation is more common for the following considerations, we shall adopt that convention in what follows.

Okay, now that we have the gauge-fixed field equation, we need to find the solution. Luckily, this differential equation is rather straightforward since it utilizes the flat spacetime wave operator $\Box = -\partial_t^2 + \vec{\nabla}^2$. This differential equation can be solved by the Green function method. A Green function $G(x, y)$ is a function that satisfies

$$\Box G(x, x') = \delta^{(4)}(x - x')\,. \tag{127}$$

Once this function is found, the general solution to (126) is given by

$$\hat{h}_{\mu\nu}(x) = \hat{h}^0_{\mu\nu}(x) - 2\kappa^2 \int \mathrm{d}^4 x'\, G(x, x') T_{\mu\nu}(x')\,. \tag{128}$$

Here, $\hat{h}^0_{\mu\nu}(x)$ is a homogeneous solution, that is, $\Box \hat{h}^0{}_{\mu\nu} = 0$, and it can be used to implement the boundary conditions. In our case it can be set to zero. Since flat spacetime is translation invariant, the function $G(x, y)$ needs to be a function of the distance between two points, $G(x, x') = G(x - x')$. Its form can be derived by various methods, and it takes the form [3]

$$G(x - x') = \frac{1}{4\pi|\vec{x} - \vec{x}'|} \delta\left(t - t' - |\vec{x} - \vec{x}'|\right) \theta(t - t')\,. \tag{129}$$

Inserting (129) into our solution (128) we find

$$\begin{aligned} \hat{h}_{\mu\nu}(x) &= -\frac{16\pi G}{4\pi} \int \mathrm{d}t' \int \mathrm{d}^3 x'\, \frac{1}{4\pi|\vec{x} - \vec{x}'|} \delta\left(t - t' - |\vec{x} - \vec{x}'|\right) \theta(t - t')\, T_{\mu\nu}(x') \\ &= -4G \int \mathrm{d}^3 x'\, \frac{1}{|\vec{x} - \vec{x}'|} T_{\mu\nu}(t - |\vec{x} - \vec{x}'|, \vec{x}')\,. \end{aligned} \tag{130}$$

The expression $t - |\vec{x} - \vec{x}'|$ is called *retarded time* and represents the fact that physical field propagate with the speed of light in flat spacetime. It is possible to show that the above form indeed satisfies the Lorenz gauge $\partial^\mu \hat{h}_{\mu\nu} = 0$, making use of $\partial^\mu T_{\mu\nu} = 0$.

3.2 The quadrupole formula

Next, let us sketch the idea how to derive the well-known *quadrupole formula*. This part will be omitting a few details here and there, since the derivation takes some detailed algebra, but I encourage you to look it up in the literature; Ch. 17.8 in [3] has a detailed derivation.

To begin with, notice that the above expression can be further simplified if we only care about the field far away from the source, that is, $|\vec{x}| \gg |\vec{x}\,'|$. To leading order in such a multipole expansion, we find

$$\hat{h}_{\mu\nu}(x) \approx -\frac{4G}{|\vec{x}|} \int \mathrm{d}^3x'\, T_{\mu\nu}(t - |\vec{x} - \vec{x}\,'|, \vec{x}\,')\,. \tag{131}$$

And last, if the particles sourcing the gravitational field move slowly compared to the speed of light (and that is typically justified in the EMRI scenario) then we can neglect retardation effects and approximate $t - |\vec{x} - \vec{x}\,'| \approx t - |\vec{x}|$ to obtain

$$\hat{h}_{\mu\nu}(x) \approx -\frac{4G}{|\vec{x}|} \int \mathrm{d}^3x'\, T_{\mu\nu}(t - |\vec{x}|, \vec{x}\,')\,. \tag{132}$$

If we now switch to a coordinate system that is tied to the center of mass, we can derive [3]

$$\hat{h}_{tt} = -\frac{4GM_{\text{tot}}}{|\vec{x}|}\,, \quad \hat{h}_{ti} = 0\,, \tag{133}$$

where $M_{\text{tot}} = \int \mathrm{d}^3x'\, T_{tt}$ is the the total mass in that coordinate system, and $\int \mathrm{d}^3x'\, T_{ti} = 0$ because we are in the center of mass frame. The purely spatial part reads as before,

$$\hat{h}_{ij}(x) = -\frac{4G}{|\vec{x}|} \int \mathrm{d}^3x'\, T_{ij}(t - |\vec{x}|, \vec{x}\,')\,. \tag{134}$$

There is a useful identity for integrals over the energy-momentum tensor going back to von Laue,

$$\int \mathrm{d}^3x'\, T_{ij}(t', \vec{x}\,') = \frac{1}{2}\frac{\partial^2}{\partial t'^2} \int \mathrm{d}^3x'\, T_{tt}(t', \vec{x}\,')x'_i x'_j\,, \qquad t' \equiv t - |\vec{x}|\,. \tag{135}$$

This can be proved by employing $\partial^\mu T_{\mu\nu} = 0$, re-expressing $\partial^i \partial^j T_{ij}$ via $\partial_t^2 T_{tt}$, and integrating by parts repeatedly. This then implies

$$\hat{h}_{ij}(x) = -\frac{2G}{|\vec{x}|}\frac{\partial^2}{\partial t'^2} \int \mathrm{d}^3x'\, T_{tt}(t', \vec{x}\,')\,. \tag{136}$$

Defining the *quadrupole moment*

$$Q_{ij}(t') \equiv \int \mathrm{d}^3x'\, T_{tt}(t', \vec{x}')x'_i x'_j \tag{137}$$

the above can be conveniently summarized in the *quadrupole formula*

$$\hat{h}_{ij}(t,x) = -\frac{2G}{|\vec{x}|}\left.\frac{\mathrm{d}^2Q_{ij}(t')}{\mathrm{d}t'^2}\right|_{t'=t-|\vec{x}|}. \tag{138}$$

Next, we will use this formula to learn about gravitational wave emission from point particles.

3.3 Gravitational wave radiation from point particles

For slowly moving point particles we can approximate $T^{tt}(t,\vec{x}) = \rho(t,\vec{x})$. Denoting the particle trajectory by $x^\mu(\tau)$, the energy density is given by

$$\rho(t,\vec{y}) = m\,\delta^{(3)}(\vec{y} - \vec{x}(\tau(t)))\,. \tag{139}$$

Note that we are employing the implicit function $\tau(t)$ that gives the proper time as a function of coordinate time t. This is required since we will be interested in describing the events as witnessed by a distant observer. At the technical level, this function can usually be found easily. Inserting the above into the quadrupole moment the integration cancels and we are left with

$$Q^{ij}(t) = m\,x^i(\tau(t))x^j(\tau(t))\,. \tag{140}$$

At this point we have to import some knowledge from the textbook literature [3]. Essentially, it is possible to define a quasilocal energy density for gravitational waves at the quadratic order in perturbation theory. This quantity is averaged over a small spacetime volume (hence quasilocal), and this is required to make this quantity gauge invariant. Second, gravitational wave emission is usually described in the transverse-traceless gauge (called TT-gauge), where the gravitational perturbation is denoted by $\hat{h}^{\mathrm{TT}}_{\mu\nu}$. Note that this is a special subcase of the Lorenz gauge. Since this object is traceless, it is useful to define the traceless quadrupole moment

$$I_{ij} \equiv Q_{ij} - \frac{1}{3}Q_{ab}\eta^{ab}\eta_{ij}\,. \tag{141}$$

For a point particle this implies

$$I^{ij}(t) = m\left(x^i(\tau(t))x^j(\tau(t)) - \frac{1}{3}\eta_{ab}x^a(\tau(t))x^b(\tau(t))\eta^{ij}\right) . \tag{142}$$

Using this quantity, one can derive that the emitted energy $\mathscr{E}$ and angular momentum $\mathscr{L}$ is

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t} = -\frac{G}{5}\left\langle I^{ij\prime\prime\prime}(t)I_{ij}{}^{\prime\prime\prime}(t)\right\rangle , \tag{143}$$

$$\frac{\mathrm{d}\mathscr{L}^i}{\mathrm{d}t} = -\frac{2G}{5}\epsilon^{ijk}\eta_{km}\left\langle I^{jl\prime\prime}(t)I^{km\prime\prime\prime}(t)\right\rangle , \tag{144}$$

where the primes denote differentiation with respect to coordinate time t, and the angular brackets denote the averaging process. We have not shown this, and this result falls "from the sky" at this point in the lecture. If you want to learn how to do this, we would need to spend a few more days deriving things.

Chapter 4

Extreme-mass ratio inspirals

4.1 Application: Schwarzschild inspiral orbits

The Schwarzschild geodesic equation (99), reduced to the equatorial plane and supplemented by the conserved quantities E and L, is

$$0 = \ddot{t} + 2\tilde{\Gamma}^t{}_{tr}\frac{E}{f}\dot{r}\,, \tag{145}$$

$$0 = \ddot{r} + \tilde{\Gamma}^r{}_{tt}\frac{E^2}{f^2} + \tilde{\Gamma}^r{}_{rr}\dot{r}^2 + \tilde{\Gamma}^r{}_{\varphi\varphi}\frac{L^2}{r^4}\,, \tag{146}$$

$$0 = \ddot{\varphi} + 2\tilde{\Gamma}^\varphi{}_{r\varphi}\dot{r}\frac{L}{r^2}\,. \tag{147}$$

In order to implement the gravitational wave backreaction, we now stipulate

$$\frac{\mathrm{d}E}{\mathrm{d}t} \approx -\frac{G}{5} I^{ij\prime\prime\prime}(t) I_{ij}{}^{\prime\prime\prime}(t)\,, \tag{148}$$

$$\frac{\mathrm{d}L^z}{\mathrm{d}t} = \frac{\mathrm{d}L}{\mathrm{d}t} \approx -\frac{2G}{5}\left(I^{x\prime\prime}_k(t) I^{k\prime\prime\prime}_y(t) - I^{y\prime\prime}_k(t) I^{k\prime\prime\prime}_x(t)\right), \tag{149}$$

that is, we assumed that all energy and angular momentum loss of the binary system is felt, at leading order, by the orbiting particle; moreover, we omitted the averaging process since the orbit is slowly evolving. Next, we need to convert the time derivatives to derivatives with respect to proper time τ using

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{1}{\dot{t}}\frac{\mathrm{d}}{\mathrm{d}\tau}\,, \tag{150}$$

which needs to be inserted repeatedly to arrive at the second and third time derivative. One finds

$$\begin{aligned}
\psi' &= \frac{\dot\psi}{\dot t}\,, \\
\psi'' &= \frac{\ddot\psi}{\dot t^2} - \frac{\dot\psi\ddot t}{\dot t^3}\,, \\
\psi''' &= \frac{\dddot\psi}{\dot t^3} - 3\frac{\ddot t\ddot\psi}{\dot t^4} - 4\frac{\dot\psi\dddot t}{\dot t^4}\,,
\end{aligned} \tag{151}$$

where $\psi = \psi(\tau)$ is a function of proper time. Explicitly, the whole system then reads

$$0 = \ddot t + 2\tilde\Gamma^t{}_{tr}\frac{E}{f}\dot r\,, \tag{152}$$

$$0 = \ddot r + \tilde\Gamma^r{}_{tt}\frac{E^2}{f^2} + \tilde\Gamma^r{}_{rr}\dot r^2 + \tilde\Gamma^r{}_{\varphi\varphi}\frac{L^2}{r^4}\,, \tag{153}$$

$$0 = \ddot\varphi + 2\tilde\Gamma^\varphi{}_{r\varphi}\dot r\frac{L}{r^2}\,, \tag{154}$$

$$\dot E = -\frac{G\dot t}{5}\left(\frac{\dddot I^{ij}}{\dot t^3} - 3\frac{\ddot t\ddot I^{ij}}{\dot t^4} - 4\frac{\dot I^{ij}\dddot t}{\dot t^4}\right)\left(\frac{\dddot I_{ij}}{\dot t^3} - 3\frac{\ddot t\ddot I_{ij}}{\dot t^4} - 4\frac{\dot I_{ij}\dddot t}{\dot t^4}\right)\,, \tag{155}$$

$$\dot L = -\frac{2G\dot t}{5}\left[\left(\frac{\ddot I^x_k}{\dot t^2} - \frac{\dot I^x_k\ddot t}{\dot t^3}\right)\left(\frac{\dddot I^k_y}{\dot t^3} - 3\frac{\ddot t\ddot I^k_y}{\dot t^4} - 4\frac{\dot I^k_y\dddot t}{\dot t^4}\right) - x\leftrightarrow y\right]\,, \tag{156}$$

where now all quantities, *including* E and L, are τ-dependent. Last, we need to parametrize the quadrupole moments. This follows from the flat-spacetime coordinate identification

$$x = r\cos\varphi\,, \qquad y = r\sin\varphi\,, \tag{157}$$

and results in

$$Q^{ij}(t) = mr^2\begin{pmatrix}\cos^2\varphi & \cos\varphi\sin\varphi & 0\\ \cos\varphi\sin\varphi & \sin^2\varphi & 0\\ 0 & 0 & 0\end{pmatrix}\,, \quad Q = Q^{ij}\eta_{ij} = \mu r^2\,, \tag{158}$$

$$I^{ij}(t) = mr^2\begin{pmatrix}\cos^2\varphi - \frac13 & \cos\varphi\sin\varphi & 0\\ \cos\varphi\sin\varphi & \sin^2\varphi - \frac13 & 0\\ 0 & 0 & -\frac13\end{pmatrix}\,. \tag{159}$$

In the above, m is the mass of the orbiting black hole. Note that in units where $GM = 1$, we can write $Gm = \mu GM$, where $\mu \equiv m/M$ is the mass ratio. Hence, all radiation corrections are suppressed by a factor of μ^2—another indication of the slow timescale of orbital evolution. The above system can now be numerically integrated, and the corresponding Mathematica sheet can be found in appendix A. While this derivation is rather qualitative at this level, it is possible to verify that this corresponds to proper motion inside the now time-dependent effective potential, and indeed leads to a slow orbital decay. In the next revision of the lecture notes, snapshots for a few cases will be added for illustration.

4.2 Estimating gravitational waveforms

Now that we have obtained the inspiral orbit, we can use the quadrupole formula (138) to determine the gravitational wave signal,

$$\hat{h}_{ij}(t,x) = -\frac{2G}{|\vec{x}|}\left.\frac{\mathrm{d}^2 Q_{ij}(t')}{\mathrm{d}t'^2}\right|_{t'=t-|\vec{x}|}\,. \tag{138}$$

While for the emitted energy and angular momentum of gravitational radiation an averaging procedure $\langle\rangle$ was required, the detection of gravitational wave signals is strictly position-dependent. This means that in what follows, the spatial dependence remains important.

The propagation and polarization of gravitational waves is most conveniently described in the so-called *transverse-traceless* gauge (or TT gauge for short), which we already mentioned before in passing. The TT gauge is a subclass of the Lorenz gauge, in which we defined

$$\hat{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}\,, \quad \partial^\mu \hat{h}_{\mu\nu} = 0\,. \tag{160}$$

Further, recall that the gauge transformation of the perturbation $h_{\mu\nu}$ is

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\,, \tag{161}$$

which for the trace-reversed perturbation $\hat{h}_{\mu\nu}$ implies

$$\hat{h}_{\mu\nu} \to \hat{h}_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu - (\partial_\alpha \epsilon^\alpha)\eta_{\mu\nu}\,, \tag{162}$$

Provided $\Box \epsilon_\mu = 0$, this gauge-transformed expression still solves the same field equations. The metric perturbation within transverse-traceless gauge is denoted as $h^{\mathrm{TT}}_{\mu\nu}$, and it is defined by

$$h^{\mathrm{TT}}_{ti} = 0\,, \quad \eta^{\mu\nu} h^{\mathrm{TT}}_{\mu\nu} = 0\,, \tag{163}$$

and these conditions are implemented via an ϵ^μ that satisfies additionally $\Box\epsilon^\mu = 0$. Since the trace vanishes, we may also consider its trace-reversed perturbation, since they coincide:

$$\hat{h}^{\mathrm{TT}}_{\mu\nu} = h^{\mathrm{TT}}_{\mu\nu}\,. \tag{164}$$

Last, note that the Lorenz gauge condition $\partial^\mu \hat{h}^{\rm TT}_{\mu\nu} = 0$ implies

$$\partial_t \hat{h}^{\rm TT}_{tt} = 0 \,. \tag{165}$$

In the case of time-dependent sources (which is what we are considering) it hence implies that $\hat{h}^{\rm TT}_{tt} = 0$. Let us introduce a spatial projection tensor in the direction of the normalized spatial vector n^i via

$$P^i_j = \delta^i_j - n^i n_j \,. \tag{166}$$

Using this tensor we can construct the transverse-traveless perturbation via

$$h^{\rm TT}_{ij} = \left(P^a_j P^j_b - \frac{1}{2} P_{ij} P^{ab} \right) \hat{h}_{ab} \,. \tag{167}$$

Adapting an orthogonal spherical reference frame spanned by the 3 unit vectors $\{e_r, e_\Theta, e_\Phi\}$ with the vector n in Cartesian coordinates taking the form

$$n^i = (\sin\Theta\cos\Phi, \sin\Theta\sin\Phi, \cos\Theta) \,, \tag{168}$$

one finds in the TT gauge [1]

$$h^{\rm TT}_{ij} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & h_{\Theta\Theta} - h_{\Phi\Phi} & 2h_{\Theta\Phi} \\ 0 & 2h_{\Theta\Phi} & h_{\Phi\Phi} - h_{\Theta\Theta} \end{pmatrix} \,, \tag{169}$$

where the projected coefficients are obtained via coordinate transformation to the spherical system,

$$\begin{aligned} \hat{h}_{\Theta\Theta} &= \cos^2\Theta\big(\hat{h}_{xx}\cos^2\Phi + \hat{h}_{xy}\sin 2\Phi + \hat{h}_{yy}\sin^2\Phi\big) + \hat{h}_{zz}\sin^2\Theta - \sin 2\Theta\big(\hat{h}_{xz}\cos\Phi + \hat{h}_{yz}\sin\Phi\big) \,, \\ \hat{h}_{\Theta\Phi} &= \cos\Theta\big(- \frac{1}{2}\hat{h}_{xx}\sin 2\Phi + \hat{h}_{xy}\cos 2\Phi + \frac{1}{2}\hat{h}_{yy}\sin 2\Phi\big) + \sin\Theta\big(\hat{h}_{xz}\sin\Phi - \hat{h}_{yz}\cos\Phi\big) \,, \\ \hat{h}_{\Phi\Phi} &= \hat{h}_{xx}\sin^2\Phi - \hat{h}_{xy}\sin 2\Phi + \hat{h}_{yy}\cos^2\Phi \,. \end{aligned} \tag{170}$$

In this notation, the two gravitational wave polarizations are given by

$$h_+ = \frac{1}{2}\left(h_{\Theta\Theta} - h_{\Phi\Phi}\right) \,, \qquad h_\times = h_{\Theta\Phi} \,. \tag{171}$$

In principle, we can now evaluate the two polarizations and plot them as a function of time, given the observation point specified by (r, Θ, Φ). While this sounds straightforward, the numerical difficulty lies again in converting between coordinate time t (which is used in the quadrupole formula) and the proper time τ (used for the description of the orbit).

While the TT gauge does give rise to physical quantities related to the polarization properties of the gravitational wave, they are not gauge invariant. For this reason it is sometimes desirable to study instead a proper gauge-invariant expression derived purely from the curvature tensors. Since in our scenario we (in principle, before we started simplifying) are considering a vacuum solution, a natural object to consider is the Weyl tensor

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \left(\eta_{\mu[\rho}R_{\sigma]\nu} - \eta_{\nu[\rho}R_{\sigma]\mu}\right) + \frac{1}{3}R\eta_{\mu[\rho}\eta_{\sigma]\nu} \, . \tag{172}$$

At the level of components, one finds

$$\begin{aligned} \frac{1}{4}\left(\ddot{h}_{\Theta\Theta} - \ddot{h}_{\Phi\Phi}\right) &= -C_{t\Theta t\Theta} = -C_{t\Phi r\Phi} = -C_{r\Theta r\Theta} = -C_{t\Phi t\Phi} = +C_{r\Phi r\Phi} = +C_{t\Theta r\Theta} = +C_{r\Phi r\Phi} \, , \\ \frac{1}{2}\ddot{h}_{\Theta\Phi} &= -C_{t\Theta t\Phi} = -C_{r\Theta r\Phi} = +C_{t\Theta r\Phi} = +C_{r\Theta t\Phi} \, . \end{aligned} \tag{173}$$

At the same time, it is customary to adopt what is called a complex null tetrad spanned by the four vectors $\{\ell^\mu, n^\mu, m^\mu, \bar{m}^\mu\}$, where $\bar{m}^\mu$ is the complex conjugate of m^μ. This goes back to Newman and Penrose. All of these vectors are null, and a possible choice for flat spacetime is

$$\ell = \frac{1}{\sqrt{2}}\left(e_t + e_r\right) \, , \quad n = \frac{1}{\sqrt{2}}\left(e_t - e_r\right) \, , \quad m = \frac{1}{\sqrt{2}}\left(e_\Theta + ie_\Phi\right) \, , \tag{174}$$

where e_t, e_r, e_Θ, and e_Φ are the unit normal vectors in each coordinate direction. Expressed by the components of these vectors, the metric of flat spacetime is

$$g_{\mu\nu} = -\ell_\mu n_\nu - n_\mu \ell_\nu + m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu \, . \tag{175}$$

We can read off the non-vanishing overlaps $\ell \cdot n = -1$, $\bar{m} \cdot \bar{m} = 1$ with the rest vanishing. Now, the Weyl tensor has 10 independent components, which in this formalism are captured by 5 complex Weyl Newman–Penrose scalars. For outgoing gravitational waves, the Weyl scalar Ψ_4 is a useful quantity. It is defined as

$$\Psi_4 \equiv C_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta \, . \tag{176}$$

In our case, it takes the form

$$\Psi_4 = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\frac{1}{2}\left(h_{\Theta\Theta} - h_{\Phi\Phi}\right) + ih_{\Theta\Phi}\right] = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(h_+ + ih_\times\right) \tag{177}$$

Essentially, it is the second derivative of the metric perturbation, whose real part is given by plus-polarization, and whose imaginary part is proportional to the cross-polarization. Especially in numerical relativity, Ψ_4 is a quantity often studied at large spatial distances, and it is hence a good candidate to connect the method described in these lecture notes to the work of other disciplines in gravitational wave physics.

4.3 Schwarzschild gravitational waveforms

Having determined the inspiral orbits, we can now feed this into the machine we developed. We first start with the quadrupole moment $Q_{ij}(t - |\vec{x}|)$ and differentiate it twice with respect to coordinate time via (151) and plug this into the quadrupole formula (138). Fixing an observation point $\{r, \Theta, \Phi\}$, we then compute the components in the corresponding transverse-traceless gauge. Finally, we reorganize the components into the two polarization modes, and then we evaluate the expressions numerically.

For one data point, the current Mathematica implementation takes around 100 milliseconds, which is not very fast. Given that in units of $GM = 1$ and $\mu = 0.01$ we needed to wait until $\tau \approx 150,000$ to see any significant inspiral, using a time spacing of 0.1 would then imply an overall evaluation time of around 42 hours (for each polarization). Clearly, this needs to be optimized. To compromise, for the purpose of this lecture, we focus on a smaller window of gravitational wave signals from $\tau = 10 \ldots 1000$ which takes the computation time down to around 15 minutes per polarization; see Fig. 2 for a plot of the two polarizations.

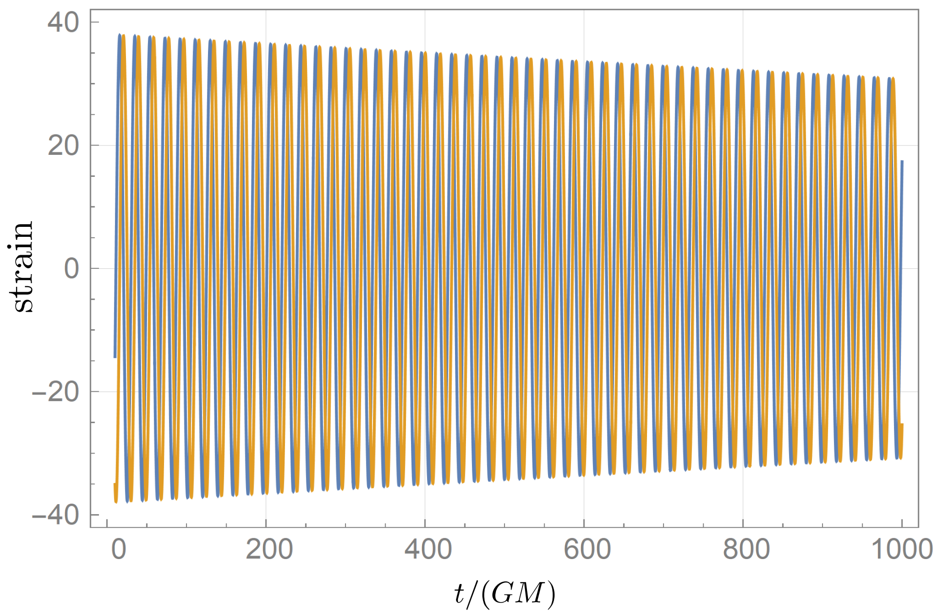

Figure 2: System parameter: $\{\mu, r_0, E, L\} = \{10^{-2}, 20GM, 0.97, 3.96GM\}$. Observation parameters: $\{r, \Theta, \Phi\} = \{10000GM, 0, 0\}$. Observation duration: $t = 10 \ldots 1000GM$. Blue: + polarization, orange: × polarization.

Chapter 5

Detecting deviations from General Relativity

One of the motivations of considering EMRIs lies in the fact that in principle they allow for long-term observations, allowing even for small deviations from general relativity to accumulate over time. In this final chapter we hence want to motivate how such a deviation could look like. While initially the plan was to focus on a black hole with torsion (hence the torsion-ful introduction to differential geometry) there is not enough time to develop this properly for the remainder of the lecture (even though in the future, when these lecture notes are extended and may be offered as a regular graduate-level course, this may change). As a compromise, let us consider instead another static black hole solution encountered in general relativity: the Reissner–Nordström solution.

This solution is the gravitational field of a point charge. If the charge parameter vanishes, one again recovers the Schwarzschild metric. Astrophysically, these solutions are not realistic, since no macroscopic object would carry a large electric charge. However, in the age of dark matter searches let us entertain the possibility that the black hole instead carries a charge of some dark vector boson A_μ. For the sake of this lecture, the feasibility of this scenario is not of primary importance—rather, it gives us a mathematically straightforward starting point, without too many changes, to track the influence of a new quantity on the physics of EMRIs.

The Einstein–Maxwell system is given by

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \,, \tag{178}$$

$$T_{\mu\nu} = F_{\mu\alpha} F_{\nu}{}^{\alpha} - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g_{\mu\nu} \,, \tag{179}$$

$$\frac{1}{\sqrt{-g}} \partial^\mu \left(\sqrt{-g} F^{\mu\nu} \right) = 0 \,, \tag{180}$$

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0\,. \tag{181}$$

Here, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor, and A_μ is the potential. The following metric and potential are an exact solution, called the *Reissner-Nordström* metric.

$$\begin{aligned} &\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2\,, \quad f(r) = 1 - \frac{2GM}{r} + \frac{q^2}{r^2}\,, \\ &A = A_\mu \mathrm{d}x^\mu = \frac{q}{r}\mathrm{d}t\,. \end{aligned} \tag{182}$$

Alternatively, if the "dark charge" is not too convincing, we may think of the line element as the source of a magnetic monopole charge p as follows [4]:

$$\begin{aligned} &\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2\,, \quad f(r) = 1 - \frac{2GM}{r} + \frac{p^2}{r^2}\,, \\ &A = A_\mu \mathrm{d}x^\mu = p\cos\theta\mathrm{d}\theta\,. \end{aligned} \tag{183}$$

Remarkably, the metric has the same form (under the substitution $g \leftrightarrow p$). In either case, notice that this geometry is still time-independent and spherically symmetric, and hence has the same Killing vectors as the Schwarzschild metric. We can even use the identical formula for the Christoffel symbols (69). We find the following effective potential and constants of motion:

$$V_{\text{eff}} = \left(1 - \frac{2GM}{r} + \frac{q^2}{r^2}\right)\left(1 + \frac{L^2}{r^2}\right)\,, \qquad E = \left(1 - \frac{2GM}{r} + \frac{q^2}{r^2}\right)\dot{t}\,, \qquad L = r^2\dot{\varphi}\,. \tag{184}$$

A detailed analytical analysis of this effective potential is more involved due to the quartic nature of the involved polynomials—for this reason we will not develop this further, but rather assume that for small values of the charge parameter q the far-distance dynamics are essentially identical to the case of $q = 0$. In particular, we impose identical boundary conditions for the motion of the secondary.

For our numerical code, we choose $q = 0.01GM$ for an initial radial distance of $r_0 = 20GM$, corresponding to a change in the metric of order $(0.01/20)^2 \sim 3 \times 10^{-7}$ at the initial location of the secondary. The resulting gravitational waveforms for the time period of $t = 0 \ldots 50GM$ can be seen in Figs. 3 and 4. Their difference is plotted in Fig. 5.

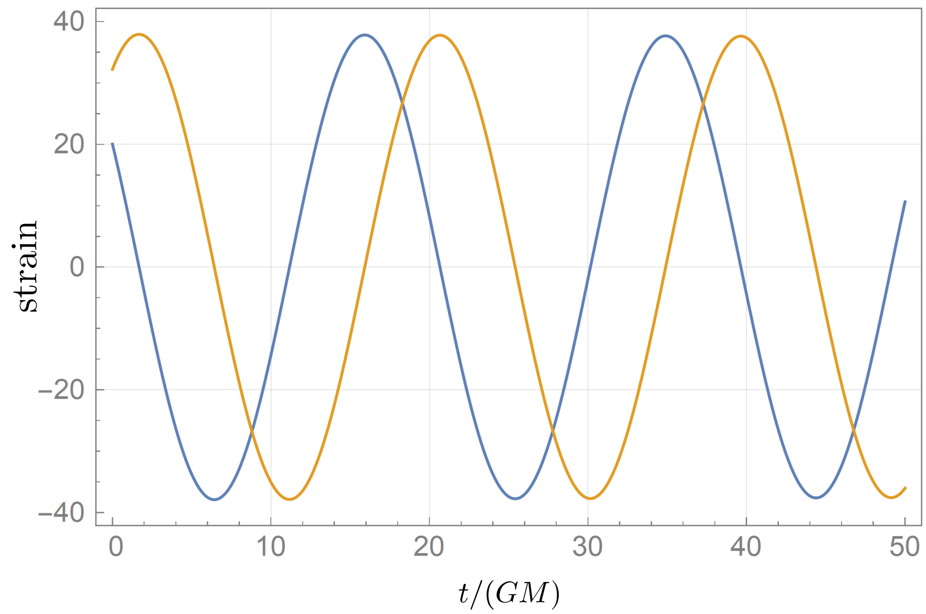

Figure 3: System parameters: $\{\mu, r_0, E, L\} = \{10^{-2}, 20GM, 0.97, 3.96GM\}$. Observation parameters: $\{r, \Theta, \Phi\} = \{10000GM, 0, 0\}$. Observation duration: $t = 10 \ldots 1000GM$. Blue: + polarization, orange: × polarization.

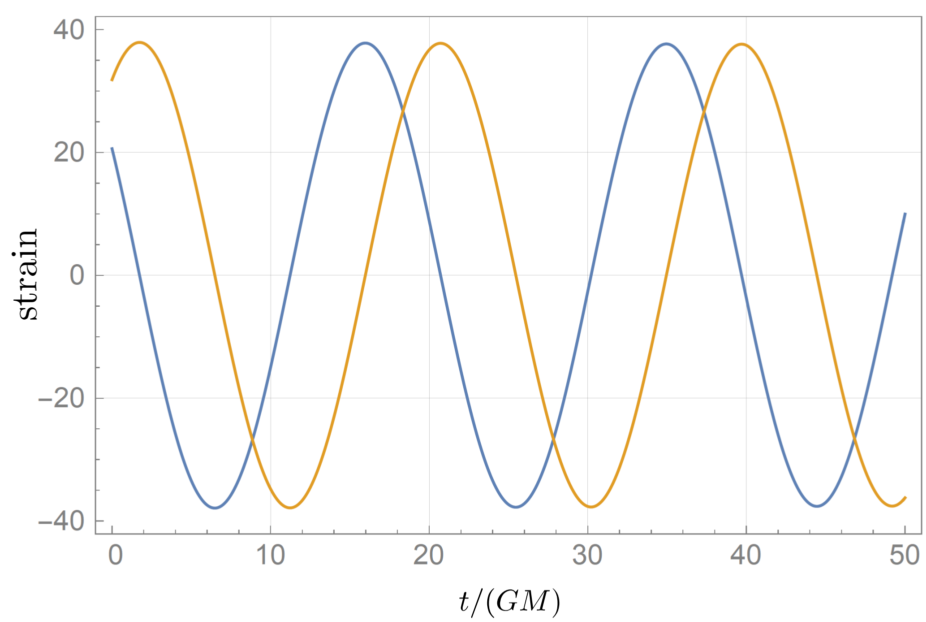

Figure 4: System parameters: $\{\mu, r_0, E, L, q\} = \{10^{-2}, 20GM, 0.97, 3.96GM, 10^{-2}\,GM\}$. Observation parameters: $\{r, \Theta, \Phi\} = \{10000GM, 0, 0\}$. Observation duration: $t = 10 \ldots 1000GM$. Blue: + polarization, orange: × polarization.

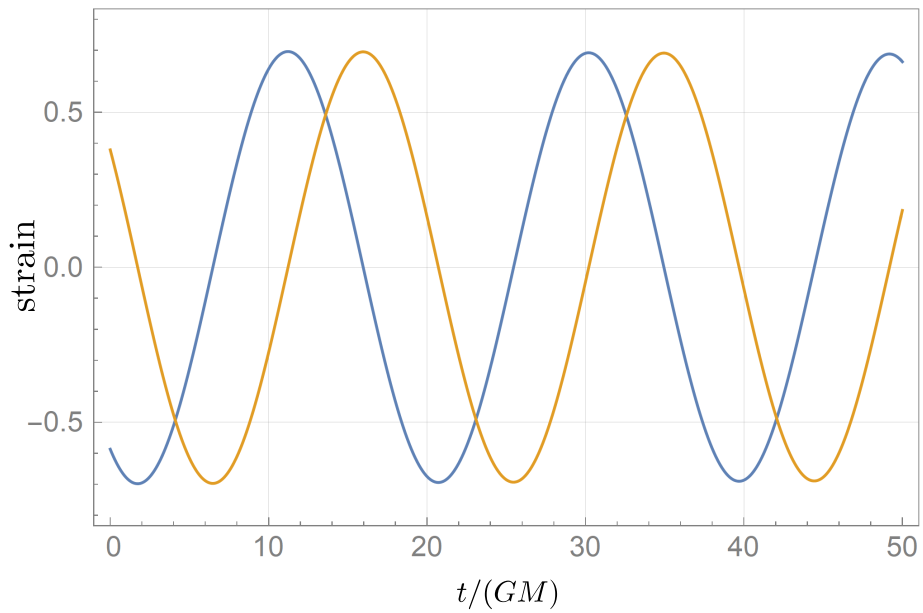

Figure 5: Difference of the two waveforms plotted in Figs. 3 and 4. The large degree of periodicity with almost equal frequency, but the much lower strain amplitude (a factor of ~ 80) suggests that the main difference is a small dephasing between the waveforms.

Conclusions

We have arrived at our goal: a MATHEMATICA sheet that computes the gravitational waveforms for an inspiral orbit, using the quadrupole formula for energy and angular momentum emission as an approximation—and you can download all sheets via the links in appendix A. Change some parameters, add retardation effects, model the self force, and see what happens!

Of course this is not the end of the story: we made several simplifications, and, most notably, we neglected important effects, such as gravitational self force. Nevertheless, as a zeroth order approximation it is still surprising that the results are *qualitatively* physically accurate.

I hope that this lecture (or these lecture notes) have inspired you to learn more about this fascinating and timely topic. Gravitational waves are a new medium that Nature has gifted us to learn more about our Universe—in a hundred years from now, will we look at gravitational waves the similarly as we look at electromagnetic waves today? Only time will tell.

Appendix A

List of Mathematica code

Course website: http://www.spintwo.net/Courses/EMRI-101/

Mathematica notebooks:

C1. Static & spherically symmetric Killing vectors
http://www.spintwo.net/Courses/EMRI-101/emri-101-c1-killing-vectors-v1.nb

C2. Schwarzschild metric: check of field equations
http://www.spintwo.net/Courses/EMRI-101/emri-101-c2-schwarzschild-derivation-v1.nb

C3. Schwarzschild metric: geodesic motion
http://www.spintwo.net/Courses/EMRI-101/emri-101-c3-schwarzschild-geodesics-v1.nb

C4. Schwarzschild metric: inspiral orbits
http://www.spintwo.net/Courses/EMRI-101/emri-101-c4-schwarzschild-inspirals-v1.nb

C5. Schwarzschild metric: gravitational waveforms
http://www.spintwo.net/Courses/EMRI-101/emri-101-c5-schwarzschild-waveforms-v1.nb

C6. Reissner–Nordström metric: gravitational waveforms
http://www.spintwo.net/Courses/EMRI-101/emri-101-c6-reissner-nordstroem-waveforms-v1.nb

C7. Comparison of waveforms
http://www.spintwo.net/Courses/EMRI-101/emri-101-c7-comparison-v1.nb

Appendix B

List of abbreviations and symbols

EMRI	extreme mass-ratio inspiral
$\equiv$	definition
M	primary black hole mass (the big, central one)
m	secondary black hole mass (the small one that is inspiralling)
μ	mass ratio $\mu = m/M \ll 1$
$\kappa = \sqrt{8\pi G}$	Einstein's gravitational constant, where G is Newton's gravitational constant
$(\mu\nu)$	symmetrization of indices, $T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$
$[\mu\nu]$	antisymmetrization of indices, $T_{[\mu\nu]} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu})$

References

[1] S. Babak, H. Fang, J. R. Gair, K. Glampedakis and S. A. Hughes, "'Kludge' gravitational waveforms for a test-body orbiting a Kerr black hole," Phys. Rev. D **75**, 024005 (2007) [Erratum: Phys. Rev. D **77**, 04990 (2008)], gr-qc/0607007. (Cited on pages 3 and 38.)

[2] A. Hanselman, "Investigating the influence of spin-curvature coupling on extreme mass-ratio inspirals," (S.B. Thesis, MIT, 2020, 49 pages, https://hdl.handle.net/1721.1/127101). (Cited on page 4.)

[3] M. P. Hobson, G. P. Efstathiou, and A. N. Lasenby, "General Relativity—An Introduction for Physicists," (Cambridge University Press, 2012). (Cited on pages 28, 31, 32, and 33.)

[4] J. Boos, "Plebański–Demiański solution of general relativity and its expressions quadratic and cubic in curvature: Analogies to electromagnetism," Int. J. Mod. Phys. D **24**, 1550079 (2015); 1412.1958 [gr-qc]. (Cited on page 43.)